

YANGIANS AND MICKELSSON ALGEBRAS I

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Abstract. We study the composition of the functor from the category of modules over the Lie algebra \mathfrak{gl}_m to the category of modules over the degenerate affine Hecke algebra of GL_N introduced by I. Cherednik, with the functor from the latter category to the category of modules over the Yangian $Y(\mathfrak{gl}_n)$ due to V. Drinfeld. We propose a representation theoretic explanation of a link between the intertwining operators on the tensor products of $Y(\mathfrak{gl}_n)$ -modules, and the “extremal cocycle” on the Weyl group of \mathfrak{gl}_m defined by D. Zhelobenko. We also establish a connection between the composition of the functors, and the “centralizer construction” of the Yangian $Y(\mathfrak{gl}_n)$ discovered by G. Olshanski.

0. Introduction

The central role in this article is played by two well known constructions. One of these constructions is due to V. Drinfeld [D2]. Let \mathfrak{H}_N be the degenerate affine Hecke algebra corresponding to the general linear group GL_N over a non-Archimedean local field. This is an associative algebra over the complex field \mathbb{C} which contains the symmetric group ring $\mathbb{C} \mathfrak{S}_N$ as a subalgebra. Let $Y(\mathfrak{gl}_n)$ be the Yangian of the general linear Lie algebra \mathfrak{gl}_n . This Yangian is a Hopf algebra over the field \mathbb{C} which contains the universal enveloping algebra $U(\mathfrak{gl}_n)$ as a subalgebra. In [D2] for any \mathfrak{H}_N -module W , an action of the algebra $Y(\mathfrak{gl}_n)$ was defined on the vector space $(W \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N}$ of the diagonal \mathfrak{S}_N -invariants in the tensor product of the vector spaces W and $(\mathbb{C}^n)^{\otimes N}$. Details of this construction are reproduced in Section 1 of the present article.

The other construction that we use here is due to I. Cherednik [C2], it was also studied by T. Arakawa, T. Suzuki and A. Tsuchiya [AST]. For any module V over the Lie algebra \mathfrak{gl}_m , it provides an action of the algebra \mathfrak{H}_N on the tensor product of \mathfrak{gl}_m -modules $V \otimes (\mathbb{C}^m)^{\otimes N}$. This action of \mathfrak{H}_N commutes with the diagonal action of \mathfrak{gl}_m on the tensor product. Details of this construction are also reproduced in Section 1 of the present article. By applying the construction from [D2] to the \mathfrak{H}_N -module $W = V \otimes (\mathbb{C}^m)^{\otimes N}$, we get an action of the Yangian $Y(\mathfrak{gl}_n)$ on

$$(V \otimes (\mathbb{C}^m)^{\otimes N} \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N} = V \otimes S^N(\mathbb{C}^m \otimes \mathbb{C}^n)$$

commuting with the action of the Lie algebra \mathfrak{gl}_m ; see our Proposition 1.3. By taking the direct sum over $N = 0, 1, 2, \dots$ of these $Y(\mathfrak{gl}_n)$ -modules, we turn into an $Y(\mathfrak{gl}_n)$ -module the vector space $V \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n)$. It is also a \mathfrak{gl}_m -module; we denote this bimodule by

$\mathcal{E}_m(V)$. The additive group \mathbb{C} acts on the Hopf algebra $Y(\mathfrak{gl}_n)$ by automorphisms. We denote by $\mathcal{E}_m^z(V)$ the $Y(\mathfrak{gl}_n)$ -module obtained from $\mathcal{E}_m(V)$ via pull-back through the automorphism of $Y(\mathfrak{gl}_n)$ corresponding to the element $z \in \mathbb{C}$. As a \mathfrak{gl}_m -module $\mathcal{E}_m^z(V)$ coincides with $\mathcal{E}_m(V)$. In this article, we identify the symmetric algebra $S(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of polynomial functions on the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$.

Now take the Lie algebra \mathfrak{gl}_{m+l} . Let \mathfrak{p} be the maximal parabolic subalgebra of \mathfrak{gl}_{m+l} containing the direct sum of Lie algebras $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$. Let \mathfrak{q} be the Abelian subalgebra of \mathfrak{gl}_{m+l} such that $\mathfrak{gl}_{m+l} = \mathfrak{q} \oplus \mathfrak{p}$. For any \mathfrak{gl}_l -module U let $V \boxtimes U$ be the \mathfrak{gl}_{m+l} -module parabolically induced from the $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ -module $V \otimes U$. This is a module induced from the subalgebra \mathfrak{p} . Consider the space $\mathcal{E}_{m+l}(V \boxtimes U)_{\mathfrak{q}}$ of \mathfrak{q} -coinvariants of the \mathfrak{gl}_{m+l} -module $\mathcal{E}_{m+l}(V \boxtimes U)$. This space is an $Y(\mathfrak{gl}_n)$ -module, which also inherits the action of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$. Our Theorem 2.1 states that the bimodule $\mathcal{E}_{m+l}(V \boxtimes U)_{\mathfrak{q}}$ over $Y(\mathfrak{gl}_n)$ and $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ is equivalent to the tensor product $\mathcal{E}_m(V) \otimes \mathcal{E}_l^m(U)$. Here we use the comultiplication on $Y(\mathfrak{gl}_n)$. The correspondence $V \mapsto \mathcal{E}_m(V)$ for $m = 1, 2, \dots$ was studied by T. Arakawa [A], but this result seems to be new. Unlike in [A], in the proof of Theorem 2.1 we do not use the representation theory of affine Hecke algebras.

In Section 3 we propose a representation theoretic explanation of the correspondence between intertwining operators on the tensor products of certain $Y(\mathfrak{gl}_n)$ -modules, and the “extremal cocycle” on the Weyl group \mathfrak{S}_m of the reductive Lie algebra \mathfrak{gl}_m , defined by D. Zhelobenko [Z1]. This correspondence, discovered by V. Tarasov and A. Varchenko [TV2], was one of the motivations of our work. The arguments of [TV2], inspired by the results of V. Toledano-Laredo [T], are based on the classical duality theorem [H] which asserts that the images of $U(\mathfrak{gl}_m)$ and $U(\mathfrak{gl}_n)$ in the ring $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of differential operators on $\mathbb{C}^m \otimes \mathbb{C}^n$ with polynomial coefficients are the commutants of each other. Relevant results were obtained by Y. Smirnov and V. Tolstoy [ST]. Our explanation of the correspondence is based on the theory of Mickelsson algebras [M1] developed in [KO]. Consider the algebra

$$U(\mathfrak{gl}_m) \otimes \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n). \quad (0.1)$$

We have a representation $\gamma : U(\mathfrak{gl}_m) \rightarrow \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Taking the composition of the comultiplication map on $U(\mathfrak{gl}_m)$ with the homomorphism $\text{id} \otimes \gamma$ we get an embedding of $U(\mathfrak{gl}_m)$ into the algebra (0.1). Our particular Mickelsson algebra is determined by the pair formed by the algebra (0.1), and its subalgebra $U(\mathfrak{gl}_m)$ relative to this embedding. From another perspective, connections between the representation theory of the Yangian $Y(\mathfrak{gl}_n)$ and the theory of Mickelsson algebras have been studied by A. Molev [M].

Our results can be restated in the language of dynamical Weyl groups as used by P. Etingof and A. Varchenko in [EV]. However, our approach makes more natural the appearance of the Yangian $Y(\mathfrak{gl}_n)$ in the context of the classical dual pair $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ of reductive Lie algebras. Moreover, results of the present article can be extended to other reductive dual pairs [H]. This will be done in our forthcoming publications.

We complete this article with an observation on the “centralizer construction” of the Yangian $Y(\mathfrak{gl}_n)$ due to G. Olshanski [O1]. For any two irreducible polynomial modules V and V' over the Lie algebra \mathfrak{gl}_m , the results of [O1] provide an action of $Y(\mathfrak{gl}_n)$ on the vector space

$$\text{Hom}_{\mathfrak{gl}_m}(V', V \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n)).$$

Moreover, this action is irreducible. Our Proposition 4.3 states that the same action is inherited from the bimodule $\mathcal{E}_m(V) = V \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n)$ over $Y(\mathfrak{gl}_n)$ and \mathfrak{gl}_m .

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1. Drinfeld functor

We begin with recalling two well known constructions from the representation theory of the *degenerate affine Hecke algebra* \mathfrak{H}_N , which corresponds to the general linear group GL_N over a local non-Archimedean field. This algebra was introduced by V. Drinfeld [D2], see also [L]. By definition, the complex associative algebra \mathfrak{H}_N is generated by the symmetric group algebra $\mathbb{C}\mathfrak{S}_N$ and by the pairwise commuting elements x_1, \dots, x_N with the cross relations for $p = 1, \dots, N-1$ and $q = 1, \dots, N$

$$\sigma_p x_q = x_q \sigma_p, \quad q \neq p, p+1; \quad (1.1)$$

$$\sigma_p x_p = x_{p+1} \sigma_p - 1. \quad (1.2)$$

Here and in what follows $\sigma_p \in \mathfrak{S}_N$ denotes the transposition of numbers p and $p+1$. More generally, $\sigma_{pq} \in \mathfrak{S}_N$ will denote the transposition of the numbers p and q . The group algebra $\mathbb{C}\mathfrak{S}_N$ can be then regarded as a subalgebra in \mathfrak{H}_N . Furthermore, it follows from the defining relations of \mathfrak{H}_N that a homomorphism $\mathfrak{H}_N \rightarrow \mathbb{C}\mathfrak{S}_N$, identical on the subalgebra $\mathbb{C}\mathfrak{S}_N \subset \mathfrak{H}_N$, can be defined by the assignments

$$x_p \mapsto \sigma_{1p} + \dots + \sigma_{p-1,p} \quad \text{for } p = 1, \dots, N. \quad (1.3)$$

We will also use the elements of the algebra \mathfrak{H}_N ,

$$y_p = x_p - \sigma_{1p} - \dots - \sigma_{p-1,p} \quad \text{where } p = 1, \dots, N.$$

Notice that $y_p \mapsto 0$ under the homomorphism $\mathfrak{H}_N \rightarrow \mathbb{C}\mathfrak{S}_N$, defined by (1.3). For any permutation $\sigma \in \mathfrak{S}_N$, we have

$$\sigma y_p \sigma^{-1} = y_{\sigma(p)}. \quad (1.4)$$

It suffices to verify (1.4) when $\sigma = \sigma_q$ and $q = 1, \dots, N-1$. Then (1.4) is equivalent to the relations (1.1), (1.2). The elements y_1, \dots, y_N do not commute, but satisfy the commutation relations

$$[y_p, y_q] = \sigma_{pq} \cdot (y_p - y_q). \quad (1.5)$$

Let us verify the equality in (1.5). Both sides of (1.5) are antisymmetric in p and q , so it suffices to consider only the case when $p < q$. Then

$$[y_p, x_q] = [x_p - \sigma_{1p} - \dots - \sigma_{p-1,p}, x_q] = 0,$$

$$[y_p, y_q] = [y_p, y_q - x_q] = -[y_p, \sigma_{1q} + \dots + \sigma_{q-1,q}] = -[y_p, \sigma_{pq}] = \sigma_{pq} \cdot (y_p - y_q)$$

where we used (1.4). Obviously, the algebra \mathfrak{H}_N is generated by $\mathbb{C}\mathfrak{S}_N$ and the elements y_1, \dots, y_N . Together with relations in $\mathbb{C}\mathfrak{S}_N$, (1.4) and (1.5) are defining relations for \mathfrak{H}_N . For more details on this presentation of the algebra \mathfrak{H}_N see [N1, Section 5].

The first construction we recall here is due to I. Cherednik [C2, Example 2.1]. It was further studied by T. Arakawa, T. Suzuki and A. Tsuchiya [AST, Section 5.3]. Let V be any module over the complex general linear Lie algebra \mathfrak{gl}_m . Let $E_{ab} \in \mathfrak{gl}_m$ with $a, b = 1, \dots, m$ be the standard matrix units. We will also regard the matrix units E_{ab} as elements of the algebra $\text{End}(\mathbb{C}^m)$, this should not cause any confusion. Let us consider the tensor product $V \otimes (\mathbb{C}^m)^{\otimes N}$ of \mathfrak{gl}_m -modules. Here each of the N tensor factors \mathbb{C}^m is a copy of the natural \mathfrak{gl}_m -module. We will use the indices $1, \dots, N$ to label these N tensor factors. For any $p = 1, \dots, N$ denote by $E_{ab}^{(p)}$ the operator on the vector space $(\mathbb{C}^m)^{\otimes N}$ acting as

$$\text{id}^{\otimes (p-1)} \otimes E_{ab} \otimes \text{id}^{\otimes (N-p)}.$$

Proposition 1.1. (i) *An action of the algebra \mathfrak{H}_N on the vector space $V \otimes (\mathbb{C}^m)^{\otimes N}$ can be defined as follows: the group $\mathfrak{S}_N \subset \mathfrak{H}_N$ acts naturally by permutations of the N tensor factors \mathbb{C}^m , while any element $y_p \in \mathfrak{H}_N$ acts as*

$$\sum_{a,b=1}^m E_{ba} \otimes E_{ab}^{(p)}. \quad (1.6)$$

(ii) *This action of \mathfrak{H}_N commutes with the (diagonal) action of \mathfrak{gl}_m on $V \otimes (\mathbb{C}^m)^{\otimes N}$.*

Proof. The relation (1.4) is obviously satisfied in this representation of the algebra \mathfrak{H}_N by operators on $V \otimes (\mathbb{C}^m)^{\otimes N}$. To verify the relation (1.5) in this representation, let us denote by Y_p the operator (1.6). For any indices $p, q = 1, \dots, N$ such that $p \neq q$, then

$$\begin{aligned} [Y_p, Y_q] &= \sum_{a,b,c,d=1}^m [E_{ba} \otimes E_{ab}^{(p)}, E_{dc} \otimes E_{cd}^{(q)}] = \sum_{a,b,c,d=1}^m (\delta_{ad} E_{bc} - \delta_{bc} E_{da}) \otimes E_{ab}^{(p)} E_{cd}^{(q)} \\ &= \sum_{a,b,c=1}^m E_{bc} \otimes E_{ab}^{(p)} E_{ca}^{(q)} - \sum_{a,b,d=1}^m E_{da} \otimes E_{ab}^{(p)} E_{bd}^{(q)}. \end{aligned}$$

Multiplying on the left the sum over a, b, c in the last line by the operator on $V \otimes (\mathbb{C}^m)^{\otimes N}$ corresponding to the transposition $\sigma_{pq} \in \mathfrak{S}_N$ results in exchanging the first indices a and c in the factors $E_{ab}^{(p)}$ and $E_{ca}^{(q)}$ in every summand. Since

$$\sum_{a=1}^m E_{aa}^{(q)} = \text{id},$$

the resulting sum equals

$$\sum_{b,c=1}^m E_{bc} \otimes E_{cb}^{(p)} = Y_p.$$

Similarly, by multiplying on the left the sum over a, b, d by the operator corresponding to $\sigma_{pq} \in \mathfrak{S}_N$ we get the operator Y_q . This completes the proof of part (i).

The proof of part (ii) consists of a direct verification that for any $p = 1, \dots, N$ and $c, d = 1, \dots, m$ the operator (1.6) commutes with the sum

$$E_{cd} \otimes \text{id} + \text{id} \otimes (E_{cd}^{(1)} + \dots + E_{cd}^{(N)})$$

which describes the action of element $E_{cd} \in \mathfrak{gl}_m$ on $V \otimes (\mathbb{C}^m)^{\otimes N}$. Here we omit the verification. Commutation of the actions of \mathfrak{S}_N and \mathfrak{gl}_m on $V \otimes (\mathbb{C}^m)^{\otimes N}$ is obvious. \square

Let us now consider the *triangular decomposition* of the Lie algebra \mathfrak{gl}_m ,

$$\mathfrak{gl}_m = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}'. \quad (1.7)$$

Here \mathfrak{h} is the Cartan subalgebra of \mathfrak{gl}_m with the basis vectors E_{11}, \dots, E_{mm} . Further, \mathfrak{n} and \mathfrak{n}' are the nilpotent subalgebras spanned respectively by the elements E_{ba} and E_{ab} for all $a, b = 1, \dots, m$ such that $a < b$. For any \mathfrak{gl}_m -module W , we denote by $W_{\mathfrak{n}}$ the vector space $W/\mathfrak{n} \cdot W$ of the coinvariants of the action of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$ on W . Note that the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$ acts on the vector space $W_{\mathfrak{n}}$.

Now consider the tensor product $W = V \otimes (\mathbb{C}^m)^{\otimes N}$ as a left module over the algebra \mathfrak{H}_N . The action of \mathfrak{H}_N on this module commutes with the action of the Lie algebra \mathfrak{gl}_m , and hence with the action of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$. So the space $(V \otimes (\mathbb{C}^m)^{\otimes N})_{\mathfrak{n}}$ of coinvariants of the action of \mathfrak{n} is a quotient of the \mathfrak{H}_N -module $V \otimes (\mathbb{C}^m)^{\otimes N}$. Thus we have a functor from the category of all \mathfrak{gl}_m -modules to the category of bimodules over \mathfrak{h} and \mathfrak{H}_N ,

$$V \mapsto (V \otimes (\mathbb{C}^m)^{\otimes N})_{\mathfrak{n}}. \quad (1.8)$$

This is a special case of the general construction due to A. Zelevinsky [Z2]. Restriction of the functor (1.8) to the category \mathcal{O} of \mathfrak{gl}_m -modules [BGG] has been used by T. Arakawa and T. Suzuki [AS, S1, S2] to give algebraic proofs of the Zelevinsky conjecture [Z1] on the multiplicities of composition factors in the standard modules over the algebra \mathfrak{H}_N , and of the Rogawski conjecture [R] on the Jantzen filtration on these modules. The first of the two conjectures was initially proved by V. Ginzburg [G] using geometric methods.

Now consider the *Yangian* $Y(\mathfrak{gl}_n)$ of the general linear Lie algebra \mathfrak{gl}_n . The Yangian $Y(\mathfrak{gl}_n)$ is a deformation of the universal enveloping algebra of the polynomial current Lie algebra $\mathfrak{gl}_n[u]$ in the class of Hopf algebras, see for instance [D1]. The unital associative algebra $Y(\mathfrak{gl}_n)$ has a family of generators

$$T_{ij}^{(1)}, T_{ij}^{(2)}, \dots \quad \text{where } i, j = 1, \dots, n.$$

The defining relations for these generators can be written in terms of the formal series

$$T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)}u^{-1} + T_{ij}^{(2)}u^{-2} + \dots \in Y(\mathfrak{gl}_n)[[u^{-1}]]. \quad (1.9)$$

Here u is the formal parameter. Let v be another formal parameter. Then the defining relations in the associative algebra $Y(\mathfrak{gl}_n)$ can be written as

$$(u - v) \cdot [T_{ij}(u), T_{kl}(v)] = T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u), \quad (1.10)$$

where $i, j, k, l = 1, \dots, n$. If $n = 1$, the algebra $Y(\mathfrak{gl}_n)$ is commutative. The relations (1.10) imply that for any $z \in \mathbb{C}$, the assignments

$$\tau_z : T_{ij}(u) \mapsto T_{ij}(u - z) \quad \text{for } i, j = 1, \dots, n \quad (1.11)$$

define an automorphism τ_z of the algebra $Y(\mathfrak{gl}_n)$. Here each of the formal power series $T_{ij}(u - z)$ in $(u - z)^{-1}$ should be re-expanded in u^{-1} , and the assignment (1.11) is a correspondence between the respective coefficients of series in u^{-1} .

Now let $E_{ij} \in \mathfrak{gl}_n$ with $i, j = 1, \dots, n$ be the standard matrix units. We will also regard the matrix units E_{ij} as elements of the algebra $\text{End}(\mathbb{C}^n)$, this should not cause any confusion. The Yangian $Y(\mathfrak{gl}_n)$ contains the universal enveloping algebra $U(\mathfrak{gl}_n)$ as a subalgebra; the embedding $U(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_n)$ can be defined by the assignments

$$E_{ij} \mapsto T_{ij}^{(1)} \quad \text{for } i, j = 1, \dots, n.$$

Moreover, there is a homomorphism $\pi_n : Y(\mathfrak{gl}_n) \rightarrow U(\mathfrak{gl}_n)$ identical on the subalgebra $U(\mathfrak{gl}_n) \subset Y(\mathfrak{gl}_n)$, it can be defined by the assignments

$$\pi_n : T_{ij}^{(2)}, T_{ij}^{(3)}, \dots \mapsto 0 \quad \text{for } i, j = 1, \dots, n. \quad (1.12)$$

For further details on the definition of the algebra $Y(\mathfrak{gl}_n)$ see [MNO, Chapter 1].

The second construction we recall here is due to V. Drinfeld [D2], this construction has originally motivated his definition of the degenerate affine Hecke algebra \mathfrak{H}_N . For $p = 1, \dots, N$ denote by $E_{ij}^{(p)}$ the operator on the vector space $(\mathbb{C}^n)^{\otimes N}$ acting as

$$\text{id}^{\otimes (p-1)} \otimes E_{ij} \otimes \text{id}^{\otimes (N-p)}.$$

The group \mathfrak{S}_N acts on the tensor product $(\mathbb{C}^n)^{\otimes N}$ from the left by permutations of the N tensor factors. Let W be any \mathfrak{H}_N -module. The group \mathfrak{S}_N also acts from the left on W , via the embedding $\mathbb{C}\mathfrak{S}_N \rightarrow \mathfrak{H}_N$. Consider the subspace

$$(W \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N} \subset W \otimes (\mathbb{C}^n)^{\otimes N} \quad (1.13)$$

of invariants with respect to the diagonal action of \mathfrak{S}_N . In the next proposition we use the convention that $(y_p)^0 = 1$, the identity element of the algebra $\mathbb{C}\mathfrak{S}_N$.

Proposition 1.2. *One can define an action of the algebra $Y(\mathfrak{gl}_n)$ on the vector space $(W \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N}$ so that for any $s = 0, 1, 2, \dots$ the generator $T_{ij}^{(s+1)}$ acts as*

$$\sum_{p=1}^N (-y_p)^s \otimes E_{ij}^{(p)}. \quad (1.14)$$

Proof. As an operator on the vector space $W \otimes (\mathbb{C}^n)^{\otimes N}$, (1.14) commutes with the diagonal action of the group \mathfrak{S}_N , due to the relations (1.4) for the generators y_1, \dots, y_N of \mathfrak{H}_N . So the restriction of the operator (1.14) to the subspace (1.13) is well defined.

By substituting the sum (1.14) for every $T_{ij}^{(s+1)}$ in (1.9), we get the series in u^{-1} with the coefficients in the algebra $\mathfrak{H}_N \otimes \text{End}((\mathbb{C}^n)^{\otimes N})$,

$$\delta_{ij} \otimes \text{id} + \sum_{s=0}^{\infty} \sum_{p=1}^N (-y_p)^s u^{-s-1} \otimes E_{ij}^{(p)} = \delta_{ij} \otimes \text{id} + \sum_{p=1}^N (u + y_p)^{-1} \otimes E_{ij}^{(p)}.$$

Making the respective substitutions for $T_{ij}(u)$ and $T_{kl}(v)$ at the left hand side of the defining relations (1.10), and then cancelling the commutators with $\delta_{ij} \otimes \text{id}$ and $\delta_{kl} \otimes \text{id}$, we obtain the sum

$$\begin{aligned} (u-v) \sum_{p,q=1}^N & \left((u+y_p)^{-1} (v+y_q)^{-1} \otimes E_{ij}^{(p)} E_{kl}^{(q)} - (v+y_q)^{-1} (u+y_p)^{-1} \otimes E_{kl}^{(q)} E_{ij}^{(p)} \right) \\ &= (u-v) \sum_{p=1}^N (u+y_p)^{-1} (v+y_p)^{-1} \otimes [E_{ij}^{(p)}, E_{kl}^{(p)}] + \end{aligned} \quad (1.15)$$

$$(u-v) \sum_{\substack{p,q=1 \\ p \neq q}}^N [(u+y_p)^{-1}, (v+y_q)^{-1}] \otimes E_{ij}^{(p)} E_{kl}^{(q)}. \quad (1.16)$$

Making the substitutions at the right hand side of (1.10), and cancelling the two tensor products $\delta_{kj} \delta_{il} \otimes \text{id}$ in the resulting difference, we get

$$\sum_{p=1}^N \left((v+y_p)^{-1} - (u+y_p)^{-1} \right) \otimes (\delta_{kj} E_{il}^{(p)} - \delta_{il} E_{kj}^{(p)}) + \quad (1.17)$$

$$\sum_{p,q=1}^N \left((u+y_p)^{-1} (v+y_q)^{-1} - (v+y_p)^{-1} (u+y_q)^{-1} \right) \otimes E_{kj}^{(p)} E_{il}^{(q)}. \quad (1.18)$$

The sums (1.15) and (1.17) are equal to each other. In the sum (1.18), the summands with $p = q$ vanish. In every summand of (1.18) with $p \neq q$, the factors $E_{kj}^{(p)}$ and $E_{il}^{(q)}$ commute. Hence, by exchanging the indices p and q , the sum (1.18) equals

$$\sum_{\substack{p,q=1 \\ p \neq q}}^N \left((u+y_q)^{-1} (v+y_p)^{-1} - (v+y_q)^{-1} (u+y_p)^{-1} \right) \otimes E_{il}^{(p)} E_{kj}^{(q)}.$$

The action of the latter sum on the subspace (1.13) coincides with the action of the sum

$$\sum_{\substack{p,q=1 \\ p \neq q}}^N \left((u+y_q)^{-1} (v+y_p)^{-1} - (v+y_q)^{-1} (u+y_p)^{-1} \right) \cdot \sigma_{pq} \otimes E_{ij}^{(p)} E_{kl}^{(q)}. \quad (1.19)$$

The sum (1.16) is equal to (1.19), because for $p \neq q$ we have the relation

$$(u-v) \cdot [(u+y_p)^{-1}, (v+y_q)^{-1}] =$$

$$\left((u + y_q)^{-1} (v + y_p)^{-1} - (v + y_q)^{-1} (u + y_p)^{-1} \right) \cdot \sigma_{pq}.$$

To verify this relation, let us multiply its sides by $(u + y_p)(v + y_q)$ on the left, and by $(v + y_q)(u + y_p)$ on the right. Using the equality

$$\sigma_{pq} \cdot (v + y_q)(u + y_p) = (v + y_p)(u + y_q) \cdot \sigma_{pq},$$

then we get the relation

$$\begin{aligned} (u - v) \cdot [u + y_p, v + y_q] = \\ \left((u + y_p)(v + y_q) - (v + y_p)(u + y_q) \right) \cdot \sigma_{pq}. \end{aligned}$$

But the last relation holds true due to (1.4) and (1.5). \square

Remark. When $s = 0$, the sum (1.14) describes the action of the element $E_{ij} \in \mathfrak{gl}_n$ on the tensor product space $W \otimes (\mathbb{C}^n)^{\otimes N}$, and hence on its subspace (1.13). Here each of the N tensor factors \mathbb{C}^n is regarded as a copy of the natural \mathfrak{gl}_n -module, and the action of \mathfrak{gl}_n on W is trivial. Hence the action of the Yangian $Y(\mathfrak{gl}_n)$ on the subspace (1.13) as defined in Proposition 1.2, is compatible with the embedding $U(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_n)$. \square

Thus we obtain a functor from the category of all \mathfrak{H}_N -modules to the category of $Y(\mathfrak{gl}_n)$ -modules

$$W \mapsto (W \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N}. \quad (1.20)$$

This is the *Drinfeld functor* for the Yangian $Y(\mathfrak{gl}_n)$. Let us now apply this functor to the \mathfrak{H}_N -module $W = V \otimes (\mathbb{C}^m)^{\otimes N}$ where V is an arbitrary \mathfrak{gl}_m -module; see Proposition 1.1. The vector space of the resulting $Y(\mathfrak{gl}_n)$ -module is

$$(V \otimes (\mathbb{C}^m)^{\otimes N} \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N} = V \otimes ((\mathbb{C}^m \otimes \mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N}$$

where the group \mathfrak{S}_N acts by permutations of the N tensor factors $\mathbb{C}^m \otimes \mathbb{C}^n$. Hence the resulting vector space is

$$V \otimes S^N(\mathbb{C}^m \otimes \mathbb{C}^n) \quad (1.21)$$

where we take the N -th symmetric power of the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$. Note that the Lie algebra \mathfrak{gl}_m also acts on (1.21) as the tensor product of two \mathfrak{gl}_m -modules.

We can identify the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$ with its dual vector space, so that the standard basis vectors of $\mathbb{C}^m \otimes \mathbb{C}^n$ are identified with the corresponding coordinate functions x_{ai} where $a = 1, \dots, m$ and $i = 1, \dots, n$. The symmetric algebra $S(\mathbb{C}^m \otimes \mathbb{C}^n)$ is then identified with the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of polynomial functions on $\mathbb{C}^m \otimes \mathbb{C}^n$. The ring of differential operators on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ with polynomial coefficients will be denoted by $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Let ∂_{ai} be the partial derivation on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ corresponding to the variable x_{ai} . We can now describe the action of $Y(\mathfrak{gl}_n)$ on the vector space (1.21).

Proposition 1.3. (i) *For any $s = 0, 1, 2, \dots$ the generator $T_{ij}^{(s+1)}$ acts on the $Y(\mathfrak{gl}_n)$ -module (1.21) as the element of the tensor product $U(\mathfrak{gl}_m) \otimes \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$,*

$$\sum_{c_0, c_1, \dots, c_s=1}^m (-1)^s E_{c_1 c_0} E_{c_2 c_1} \dots E_{c_s c_{s-1}} \otimes x_{c_0 i} \partial_{c_s j}. \quad (1.22)$$

In the case $s = 0$, the first tensor factor in the summand in (1.22) is understood as 1.

(ii) *The action of $Y(\mathfrak{gl}_n)$ on (1.21) commutes with the (diagonal) action of \mathfrak{gl}_m .*

Proof. First let us consider the action of the sum (1.14) on the vector space $W \otimes (\mathbb{C}^n)^{\otimes N}$ where $W = V \otimes (\mathbb{C}^m)^{\otimes N}$. By substituting the sum (1.6) for y_p in (1.14), we then get the sum

$$\sum_{p=1}^N \left(- \sum_{a,b=1}^m E_{ba} \otimes E_{ab}^{(p)} \right)^s \otimes E_{ij}^{(p)} \quad (1.23)$$

acting on the vector space $V \otimes (\mathbb{C}^m)^{\otimes N} \otimes (\mathbb{C}^n)^{\otimes N}$. Using the relations

$$E_{ab}^{(p)} E_{cd}^{(p)} = \delta_{bc} E_{ad}^{(p)} \quad \text{for } p = 1, \dots, N$$

the sum (1.23) can be rewritten as

$$\sum_{p=1}^N \sum_{c_0, c_1, \dots, c_s=1}^m (-1)^s E_{c_1 c_0} \dots E_{c_s c_{s-1}} \otimes E_{c_0 c_s}^{(p)} \otimes E_{ij}^{(p)}.$$

To prove the part (i) of the proposition, it remains to observe that after identifying the subspace

$$((\mathbb{C}^m)^{\otimes N} \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N} \subset (\mathbb{C}^m)^{\otimes N} \otimes (\mathbb{C}^n)^{\otimes N}$$

with the space $\mathcal{P}^N(\mathbb{C}^m \otimes \mathbb{C}^n)$ of polynomial functions on $\mathbb{C}^m \otimes \mathbb{C}^n$ of degree N , the operator

$$\sum_{p=1}^N E_{c_0 c_s}^{(p)} \otimes E_{ij}^{(p)}$$

on this subspace can be identified with the operator $x_{c_0 i} \partial_{c_s j}$ on the space $\mathcal{P}^N(\mathbb{C}^m \otimes \mathbb{C}^n)$. The part (ii) of Proposition 1.3 follows from the respective part of Proposition 1.1. \square

Remark. By definition, the basis element $E_{ab} \in \mathfrak{gl}_m$ acts on the vector space (1.21) as

$$E_{ab} \otimes 1 + \sum_{k=1}^n 1 \otimes x_{ak} \partial_{bk}. \quad (1.24)$$

One can easily verify by straightforward calculation, that the elements (1.22) and (1.24) of the algebra $U(\mathfrak{gl}_m) \otimes \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ commute with each other. Moreover, using the First Fundamental Theorem of invariants for the general linear group GL_m , one can show that the commutant in the algebra $U(\mathfrak{gl}_m) \otimes \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of all elements (1.24) with $a, b = 1, \dots, m$ is generated by the subalgebra $Z(\mathfrak{gl}_m) \otimes 1$ and all elements of the form (1.22); cf. [O2, Section 2.1]. Here $Z(\mathfrak{gl}_m)$ denotes the centre of the universal enveloping algebra $U(\mathfrak{gl}_m)$. This extends the classical theorem [H, Section 2.3] stating that the two families of operators on the vector space $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$,

$$\sum_{k=1}^n x_{ak} \partial_{bk} \quad \text{where } a, b = 1, \dots, m \quad (1.25)$$

and

$$\sum_{c=1}^m x_{ci} \partial_{cj} \quad \text{where } i, j = 1, \dots, n \quad (1.26)$$

generate their mutual commutants in the algebra $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Here the operators (1.25) and (1.26) describe the actions on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of the basis elements $E_{ab} \in \mathfrak{gl}_m$ and $E_{ij} \in \mathfrak{gl}_n$ respectively. \square

We finish this section with an observation on matrices with entries from the universal enveloping algebra $U(\mathfrak{gl}_m)$. Let E be the $m \times m$ matrix whose ab -entry is the generator $E_{ab} \in \mathfrak{gl}_m$. Let E' be the transposed matrix. Take the matrix inverse to $u + E'$. Here the summand u stands for the scalar $m \times m$ matrix with diagonal entry u , and the inverse is a formal power series in u^{-1} with matrix coefficients. Denote by $X_{ab}(u)$ the ab -entry of inverse matrix. Then

$$X_{ab}(u) = \delta_{ab} u^{-1} - E_{ba} u^{-2} + \sum_{s=1}^{\infty} \sum_{c_1, \dots, c_s=1}^m (-1)^{s+1} E_{c_1 a} E_{c_2 c_1} \dots E_{c_s c_{s-1}} E_{bc_s} u^{-s-2}. \quad (1.27)$$

The assignment of the element (1.22) to any coefficient $T_{ij}^{(s+1)}$ of the series (1.9) can be now written as

$$T_{ij}(u) \mapsto \delta_{ij} + \sum_{a,b=1}^m X_{ab}(u) \otimes x_{ai} \partial_{bj}.$$

2. Parabolic induction

The Yangian $Y(\mathfrak{gl}_n)$ is a Hopf algebra over the field \mathbb{C} . Using the series (1.9), the comultiplication $\Delta : Y(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$ is defined by the assignment

$$\Delta : T_{ij}(u) \mapsto \sum_{k=1}^n T_{ik}(u) \otimes T_{kj}(u); \quad (2.1)$$

the tensor product at the right hand side of the assignment (2.1) is taken over the subalgebra $\mathbb{C}[[u^{-1}]] \subset Y(\mathfrak{gl}_n)[[u^{-1}]]$. When taking tensor products of modules over $Y(\mathfrak{gl}_n)$, we use the comultiplication (2.1). The counit homomorphism $\varepsilon : Y(\mathfrak{gl}_n) \rightarrow \mathbb{C}$ is defined by

$$\varepsilon : T_{ij}(u) \mapsto \delta_{ij} \cdot 1.$$

The antipode S on $Y(\mathfrak{gl}_n)$ is defined by using the $n \times n$ matrix $T(u)$ whose ij -entry is the series $T_{ij}(u)$. This matrix is invertible as formal power series in u^{-1} with matrix coefficients, because the leading term of this series is the identity $n \times n$ matrix. Then the involutive anti-automorphism S of $Y(\mathfrak{gl}_n)$ is defined by the assignment

$$S : T(u) \mapsto T(u)^{-1}.$$

This assignment means that by applying S to the coefficients of the series $T_{ij}(u)$, we obtain the series which is the ij -entry of the inverse matrix $T(u)^{-1}$. We also use the involutive automorphism ω_n of the algebra $Y(\mathfrak{gl}_n)$ defined by a similar assignment,

$$\omega_n : T(u) \mapsto T(-u)^{-1}. \quad (2.2)$$

For more details on the Hopf algebra structure on $Y(\mathfrak{gl}_n)$ see again [MNO, Chapter 1].

Let us now consider the infinite direct sum of bimodules over \mathfrak{gl}_m and $Y(\mathfrak{gl}_n)$,

$$\bigoplus_{N=0}^{\infty} V \otimes S^N(\mathbb{C}^m \otimes \mathbb{C}^n) = V \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n).$$

Let us denote this bimodule by $\mathcal{E}_m(V)$, so that \mathcal{E}_m is a functor from the category of all \mathfrak{gl}_m -modules to the category of bimodules over \mathfrak{gl}_m and $Y(\mathfrak{gl}_n)$. By identifying the symmetric algebra $S(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$, the action of the generator $T_{ij}^{(s+1)}$ of $Y(\mathfrak{gl}_n)$ on $\mathcal{E}_m(V)$ is described by the formula (1.22).

For any positive integer l let U be a module over the Lie algebra \mathfrak{gl}_l . Then $\mathcal{E}_l(U)$ is another $Y(\mathfrak{gl}_n)$ -module. For any $z \in \mathbb{C}$ denote by $\mathcal{E}_l^z(U)$ the $Y(\mathfrak{gl}_n)$ -module obtained from $\mathcal{E}_l(U)$ via pull-back through the automorphism τ_z of $Y(\mathfrak{gl}_n)$, defined by (1.11). As a \mathfrak{gl}_l -module $\mathcal{E}_l^z(U)$ coincides with $\mathcal{E}_l(U)$.

The decomposition $\mathbb{C}^{m+l} = \mathbb{C}^m \oplus \mathbb{C}^l$ determines an embedding of the direct sum $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ of Lie algebras into \mathfrak{gl}_{m+l} . As a subalgebra of \mathfrak{gl}_{m+l} , the direct summand \mathfrak{gl}_m is spanned by the matrix units $E_{ab} \in \mathfrak{gl}_{m+l}$ where $a, b = 1, \dots, m$. The direct summand \mathfrak{gl}_l is spanned by the matrix units E_{ab} where $a, b = m+1, \dots, m+l$. Let \mathfrak{q} and \mathfrak{q}' be the Abelian subalgebras of \mathfrak{gl}_{m+l} spanned respectively by matrix units E_{ba} and E_{ab} for all $a = 1, \dots, m$ and $b = m+1, \dots, m+l$. Put $\mathfrak{p} = \mathfrak{gl}_m \oplus \mathfrak{gl}_l \oplus \mathfrak{q}'$. Then \mathfrak{p} is a maximal parabolic subalgebra of the reductive Lie algebra \mathfrak{gl}_{m+l} , and moreover $\mathfrak{gl}_{m+l} = \mathfrak{q} \oplus \mathfrak{p}$. Denote by $V \boxtimes U$ the \mathfrak{gl}_{m+l} -module *parabolically induced* from the $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ -module $V \otimes U$. To define $V \boxtimes U$, one first extends the action of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ on $V \otimes U$ to the Lie algebra \mathfrak{p} , so that any element of the subalgebra $\mathfrak{q}' \subset \mathfrak{p}$ acts on $V \otimes U$ as zero. By definition, $V \boxtimes U$ is the \mathfrak{gl}_{m+l} -module induced from the \mathfrak{p} -module $V \otimes U$.

Now consider the bimodule $\mathcal{E}_{m+l}(V \boxtimes U)$ over \mathfrak{gl}_{m+l} and $Y(\mathfrak{gl}_n)$. Here the action of $Y(\mathfrak{gl}_n)$ commutes with the action of the Lie algebra \mathfrak{gl}_{m+l} , and hence with the action of the subalgebra $\mathfrak{q} \subset \mathfrak{gl}_{m+l}$. Therefore the vector space $\mathcal{E}_{m+l}(V \boxtimes U)_{\mathfrak{q}}$ of coinvariants of the action of the subalgebra \mathfrak{q} is a quotient of the $Y(\mathfrak{gl}_n)$ -module $\mathcal{E}_{m+l}(V \boxtimes U)$. Note that the subalgebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l \subset \mathfrak{gl}_{m+l}$ also acts on this quotient space.

Theorem 2.1. *The bimodule $\mathcal{E}_{m+l}(V \boxtimes U)_{\mathfrak{q}}$ over the Yangian $Y(\mathfrak{gl}_n)$ and the direct sum $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$, is equivalent to the tensor product $\mathcal{E}_m(V) \otimes \mathcal{E}_l^m(U)$.*

Our proof of the theorem is based on two simple lemmas. The first of these lemmas applies to matrices over arbitrary unital ring. Take a matrix of size $(m+l) \times (m+l)$ over such a ring, and write it as the block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (2.3)$$

where the blocks A, B, C, D are matrices of sizes $m \times m$, $m \times l$, $l \times m$, $l \times l$ respectively. The following fact is well known, see for instance [B, Lemma 3.2].

Lemma 2.2. *Suppose the matrix (2.3) is invertible. Suppose the matrices A and D are also invertible. Then the matrices $A - BD^{-1}C$ and $D - CA^{-1}B$ are invertible too, and*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}.$$

Consider again the $m \times m$ matrix E whose ab -entry is the generator $E_{ab} \in \mathfrak{gl}_m$. The ab -entry $X_{ab}(u)$ of the matrix inverse to $u + E'$ is given by the equality (1.27). Denote by $Z(u)$ the trace of the inverse matrix, so that

$$Z(u) = \sum_{c=1}^m X_{cc}(u). \quad (2.4)$$

Then $Z(u)$ is a formal power series in u^{-1} with the coefficients from the algebra $U(\mathfrak{gl}_m)$. Note that the leading term of this series is mu^{-1} . Let us now regard the coefficients of the series (1.27) and (2.4) as elements of the algebra $U(\mathfrak{gl}_{m+l})$, using the standard embedding of the Lie algebra \mathfrak{gl}_m to \mathfrak{gl}_{m+l} .

Lemma 2.3. *For any $a = 1, \dots, m$ and $d = 1, \dots, l$ we have an equality of the series with coefficients in the algebra $U(\mathfrak{gl}_{m+l})$,*

$$\sum_{b=1}^m E_{m+d,b} X_{ab}(u) = \sum_{b=1}^m X_{ab}(u) E_{m+d,b} (1 - Z(u)). \quad (2.5)$$

Proof. For any indices $b, c, d, e = 1, \dots, m$ we have the equality

$$(\delta_{ec} u + E_{ec}) E_{m+d,b} = E_{m+d,b} (\delta_{ec} u + E_{ec}) - \delta_{eb} E_{m+d,c}.$$

Multiplying both sides of this equality by $X_{ac}(u)$ on the left and taking the sums over $c = 1, \dots, m$ we obtain the equality

$$\delta_{ae} E_{m+d,b} = \sum_{c=1}^m X_{ac}(u) E_{m+d,b} (\delta_{ec} u + E_{ec}) - \sum_{c=1}^m X_{ac}(u) \delta_{eb} E_{m+d,c}.$$

Multiplying both sides of the latter equality by $X_{eb}(u)$ on the right and taking the sums over $e = 1, \dots, m$ we get the equality

$$E_{m+d,b} X_{ab}(u) = X_{ab}(u) E_{m+d,b} - \sum_{c=1}^m X_{ac}(u) E_{m+d,c} X_{bb}(u).$$

Taking here the sums over $b = 1, \dots, m$ and using the definition (2.4) we get (2.5). \square

Proof of Theorem 2.1. The vector space of the \mathfrak{gl}_{m+l} -module $V \boxtimes U$ can be identified with the tensor product $U(\mathfrak{q}) \otimes V \otimes U$ so that the Lie subalgebra $\mathfrak{q} \subset \mathfrak{gl}_{m+l}$ acts via left multiplication on the first tensor factor. Note that the corresponding action of the commutative algebra $U(\mathfrak{q})$ is free. The tensor product $V \otimes U$ is then identified with the subspace

$$1 \otimes V \otimes U \subset U(\mathfrak{q}) \otimes V \otimes U. \quad (2.6)$$

On this subspace, any element of the subalgebra $\mathfrak{q}' \subset \mathfrak{gl}_{m+l}$ acts as zero, while the two direct summands of subalgebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l \subset \mathfrak{gl}_{m+l}$ act non-trivially only on the tensor factors V and U respectively. All this determines the action of Lie algebra \mathfrak{gl}_{m+l} on

$U(\mathfrak{q}) \otimes V \otimes U$. Now consider $\mathcal{E}_{m+l}(V \boxtimes U)$ as a \mathfrak{gl}_{m+l} -module, we will denote it by W for short. Then W is the tensor product of two \mathfrak{gl}_{m+l} -modules,

$$W = (V \boxtimes U) \otimes \mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n) = U(\mathfrak{q}) \otimes V \otimes U \otimes \mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n).$$

The vector spaces of the two $Y(\mathfrak{gl}_n)$ -modules $\mathcal{E}_m(V)$ and $\mathcal{E}_l^m(U)$ are respectively

$$V \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n) \quad \text{and} \quad U \otimes \mathcal{P}(\mathbb{C}^l \otimes \mathbb{C}^n).$$

Identify the tensor product of these two vector spaces with

$$V \otimes U \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n) \otimes \mathcal{P}(\mathbb{C}^l \otimes \mathbb{C}^n) = V \otimes U \otimes \mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n) \quad (2.7)$$

where we use the standard direct sum decomposition

$$\mathbb{C}^{m+l} \otimes \mathbb{C}^n = \mathbb{C}^m \otimes \mathbb{C}^n \oplus \mathbb{C}^l \otimes \mathbb{C}^n.$$

Regard the tensor product $V \otimes U$ in (2.7) as a module over the subalgebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l \subset \mathfrak{gl}_{m+l}$. This subalgebra also acts on $\mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n)$ naturally. Define a linear map

$$\chi : V \otimes U \otimes \mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n) \rightarrow W / \mathfrak{q} \cdot W$$

by the assignment

$$\chi : y \otimes x \otimes f \mapsto 1 \otimes y \otimes x \otimes f + \mathfrak{q} \cdot W$$

for any $y \in V$, $x \in U$ and $f \in \mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n)$. The operator χ evidently intertwines the actions of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$.

Let us demonstrate that the operator χ is bijective. Firstly consider the action of the Lie subalgebra $\mathfrak{q} \subset \mathfrak{gl}_{m+l}$ on the vector space

$$\mathcal{P}(\mathbb{C}^{m+l}) = \mathcal{P}(\mathbb{C}^m) \otimes \mathcal{P}(\mathbb{C}^l).$$

This vector space admits an ascending filtration by the subspaces

$$\bigoplus_{N=0}^K \mathcal{P}^N(\mathbb{C}^m) \otimes \mathcal{P}(\mathbb{C}^l) \quad \text{where} \quad K = 0, 1, 2, \dots$$

Here $\mathcal{P}^N(\mathbb{C}^m)$ is the space of polynomial functions on \mathbb{C}^m of degree N . The action of the Lie algebra \mathfrak{q} on $\mathcal{P}(\mathbb{C}^{m+l})$ preserves each of these subspaces, and is trivial on the associated graded space. Similarly, the vector space $\mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n)$ admits an ascending filtration by \mathfrak{q} -submodules such that \mathfrak{q} acts trivially on each of the corresponding graded subspaces. The latter filtration induces a filtration of W by \mathfrak{q} -submodules such that the corresponding graded quotient $\text{gr } W$ is a free $U(\mathfrak{q})$ -module. The space of coinvariants $(\text{gr } W)_{\mathfrak{q}}$ is therefore isomorphic to $V \otimes U \otimes \mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n)$, via the bijective linear map

$$y \otimes x \otimes f \mapsto 1 \otimes y \otimes x \otimes f + \mathfrak{q} \cdot (\text{gr } W).$$

Therefore the linear map χ is bijective as well.

Let us now demonstrate that the map χ intertwines the actions of the Yangian $Y(\mathfrak{gl}_n)$. Consider the $(m+l) \times (m+l)$ matrix whose ab -entry is $\delta_{ab}u + E_{ba}$. Here we regard E_{ba} as an element of the algebra $U(\mathfrak{gl}_{m+l})$. Write this matrix in the block form (2.3) where A, B, C, D are matrices of sizes $m \times m$, $m \times l$, $l \times m$, $l \times l$ respectively. In the notation introduced in the end of Section 1, here $A = u + E'$. Using the observation made there along with the definition (2.1) of the comultiplication, the action of the algebra $Y(\mathfrak{gl}_n)$ on the vector space (2.7) of the tensor product of the two $Y(\mathfrak{gl}_n)$ -modules $\mathcal{E}_m(V)$ and $\mathcal{E}_l^m(U)$ can be described by assigning to every series $T_{ij}(u)$ the product of the series

$$\begin{aligned} & \sum_{k=1}^n \left(\delta_{ik} + \sum_{a,b=1}^m (A^{-1})_{ab} \otimes x_{ai} \partial_{bk} \right) \left(\delta_{kj} + \sum_{c,d=1}^l ((D-m)^{-1})_{cd} \otimes x_{m+c,k} \partial_{m+d,j} \right) \\ &= \delta_{ij} + \sum_{a,b=1}^m (A^{-1})_{ab} \otimes x_{ai} \partial_{bj} + \sum_{c,d=1}^l ((D-m)^{-1})_{cd} \otimes x_{m+c,i} \partial_{m+d,j} + \end{aligned} \quad (2.8)$$

$$\sum_{k=1}^n \sum_{a,b=1}^m \sum_{c,d=1}^l (A^{-1})_{ab} ((D-m)^{-1})_{cd} \otimes x_{ai} \partial_{bk} x_{m+c,k} \partial_{m+d,j}. \quad (2.9)$$

Note that in (2.9) we have $\partial_{bk} x_{m+c,k} = x_{m+c,k} \partial_{bk}$ because $b \leq m$. The first tensor factors of all summands in (2.8) and (2.9) correspond to the action of the universal enveloping algebra $U(\mathfrak{gl}_m \oplus \mathfrak{gl}_l)$ on $V \otimes U$.

Let us now write the matrix inverse to (2.3) as the block matrix

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$$

where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are matrices of sizes $m \times m$, $m \times l$, $l \times m$, $l \times l$ respectively. Each of these four blocks is regarded as formal power series in u^{-1} with matrix coefficients. The entries of these matrix coefficients belong to the algebra $U(\mathfrak{gl}_{m+l})$. By once again using the observation made in the end of Section 1, the action of $Y(\mathfrak{gl}_n)$ on the vector space W can now be described by assigning to every series $T_{ij}(u)$ the sum of the series

$$\begin{aligned} & \delta_{ij} + \sum_{a,b=1}^m \tilde{A}_{ab} \otimes x_{ai} \partial_{bj} + \sum_{a=1}^m \sum_{d=1}^l \tilde{B}_{ad} \otimes x_{ai} \partial_{m+d,j} + \\ & \sum_{b=1}^m \sum_{c=1}^l \tilde{C}_{cb} \otimes x_{m+c,i} \partial_{bj} + \sum_{c,d=1}^l \tilde{D}_{cd} \otimes x_{m+c,i} \partial_{m+d,j}. \end{aligned}$$

The first tensor factors in all summands here correspond to the action of the algebra $U(\mathfrak{gl}_{m+l})$ on the vector space $U(\mathfrak{q}) \otimes V \otimes U$ of the parabolically induced module $V \boxtimes U$.

Let us apply these tensor factors to elements of the subspace (2.6). By Lemma 2.2,

$$\tilde{A} = (A - BD^{-1}C)^{-1}.$$

All entries of the matrix C belong to \mathfrak{q}' and hence act on the subspace (2.6) as zeroes. Further, we have $A = u + E'$. Every entry of the matrix E' belongs to the subalgebra $\mathfrak{gl}_m \subset \mathfrak{gl}_{m+l}$ and the adjoint action of this subalgebra on \mathfrak{gl}_{m+l} preserves \mathfrak{q}' . Therefore the results of applying $(A^{-1})_{ab}$ and \tilde{A}_{ab} to elements of the subspace (2.6) are the same. Similar arguments show that any entry of the matrix

$$\tilde{C} = -D^{-1}C(A - BD^{-1}C)^{-1} = -D^{-1}C\tilde{A}$$

act on the subspace (2.6) as zero.

Consider the matrix

$$\tilde{D} = (D - CA^{-1}B)^{-1}.$$

In the notation of Lemma 2.3 the ab -entry of the matrix A^{-1} is $X_{ab}(u)$, and the trace of A^{-1} is $Z(u)$. Using that lemma, the cd -entry of the $l \times l$ matrix $D - CA^{-1}B$ equals

$$\begin{aligned} & \delta_{cd} u + E_{m+d, m+c} - \sum_{a, b=1}^m E_{a, m+c} X_{ab}(u) E_{m+d, b} = \delta_{cd} u + E_{m+d, m+c} \\ & - \sum_{a, b=1}^m E_{a, m+c} E_{m+d, b} X_{ab}(u) (1 - Z(u))^{-1} = \delta_{cd} u + E_{m+d, m+c} \\ & - \sum_{a, b=1}^m (E_{m+d, b} E_{a, m+c} + \delta_{cd} E_{ab} - \delta_{ab} E_{m+d, m+c}) X_{ab}(u) (1 - Z(u))^{-1} = \\ & \delta_{cd} (u - (m - u Z(u))(1 - Z(u))^{-1}) + E_{m+d, m+c} (1 + Z(u)(1 - Z(u))^{-1}) \end{aligned} \quad (2.10)$$

$$- \sum_{a, b=1}^m E_{m+d, b} E_{a, m+c} X_{ab}(u) (1 - Z(u))^{-1}. \quad (2.11)$$

We used the identity

$$\sum_{a, b=1}^m E_{ab} X_{ab}(u) = m - u Z(u)$$

which follows from the definitions (1.27) and (2.4). The expression in the line (2.10) is equal to

$$(D - m)_{cd} (1 - Z(u))^{-1}.$$

The factor $E_{a, m+c}$ in any summand in the line (2.11) belongs to \mathfrak{q}' while every element of \mathfrak{q}' acts on the subspace (2.6) as zero. The coefficients of the series $X_{ab}(u)$ and $Z(u)$ in (2.11) belong to $U(\mathfrak{gl}_m)$ while the adjoint action of subalgebra $\mathfrak{gl}_m \subset \mathfrak{gl}_{m+l}$ preserves \mathfrak{q}' . The adjoint action of every element $E_{m+d, m+c} \in \mathfrak{gl}_{m+l}$ also preserves \mathfrak{q}' . Therefore the result of applying \tilde{D}_{cd} to elements of the subspace (2.6) is the same as that of applying

$$(1 - Z(u))((D - m)^{-1})_{cd}.$$

Now consider

$$\tilde{B} = -A^{-1}B(D - CA^{-1}B)^{-1} = -A^{-1}B\tilde{D}.$$

The above arguments show that the result of applying the ad -entry of this matrix to elements of the subspace (2.6) is the same as that of applying the ad -entry of the matrix

$$-A^{-1}B(1 - Z(u))(D - m)^{-1}.$$

By using Lemma 2.3 once again, the latter entry equals

$$\begin{aligned} & - \sum_{b=1}^m \sum_{c=1}^l X_{ab}(u) E_{m+c,b} (1 - Z(u)) ((D - m)^{-1})_{cd} = \\ & - \sum_{b=1}^m \sum_{c=1}^l E_{m+c,b} X_{ab}(u) ((D - m)^{-1})_{cd}. \end{aligned}$$

Thus we have proved that the action of $Y(\mathfrak{gl}_n)$ on the elements of the subspace

$$1 \otimes V \otimes U \otimes \mathcal{P}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n) \subset W \quad (2.12)$$

can be described by assigning to every series $T_{ij}(u)$ the sum of the series

$$\begin{aligned} & \delta_{ij} + \sum_{a,b=1}^m (A^{-1})_{ab} \otimes x_{ai} \partial_{bj} + \sum_{c,d=1}^l (1 - Z(u)) ((D - m)^{-1})_{cd} \otimes x_{m+c,i} \partial_{m+d,j} \\ & - \sum_{a,b=1}^m \sum_{c,d=1}^l E_{m+c,b} X_{ab}(u) ((D - m)^{-1})_{cd} \otimes x_{ai} \partial_{m+d,j}. \end{aligned} \quad (2.13)$$

Let us now consider the results of the action of $Y(\mathfrak{gl}_n)$ on this subspace modulo $\mathfrak{q} \cdot W$. Since $E_{m+c,b} \in \mathfrak{q}$, the expression in the line (2.13) can be then replaced by

$$\begin{aligned} & \sum_{k=1}^n \sum_{a,b=1}^m \sum_{c,d=1}^l X_{ab}(u) ((D - m)^{-1})_{cd} \otimes x_{m+c,k} \partial_{bk} x_{ai} \partial_{m+d,j} = \\ & \sum_{k=1}^n \sum_{a,b=1}^m \sum_{c,d=1}^l (A^{-1})_{ab} ((D - m)^{-1})_{cd} \otimes x_{m+c,k} x_{ai} \partial_{bk} \partial_{m+d,j} + \\ & \sum_{c,d=1}^l Z(u) ((D - m)^{-1})_{cd} \otimes x_{m+c,i} \partial_{m+d,j}. \end{aligned}$$

Here we used the equality of differential operators $\partial_{bk} x_{ai} = x_{ai} \partial_{bk} + \delta_{ab} \delta_{ik}$. By making this replacement we show that modulo $\mathfrak{q} \cdot W$, the action of $Y(\mathfrak{gl}_n)$ on elements of the subspace (2.12) can be described by assigning to every series $T_{ij}(u)$ the sum of the series

$$\delta_{ij} + \sum_{a,b=1}^m (A^{-1})_{ab} \otimes x_{ai} \partial_{bj} + \sum_{c,d=1}^l (1 - Z(u)) ((D - m)^{-1})_{cd} \otimes x_{m+c,i} \partial_{m+d,j} +$$

$$\sum_{k=1}^n \sum_{a,b=1}^m \sum_{c,d=1}^l (A^{-1})_{ab} ((D-m)^{-1})_{cd} \otimes x_{m+c,k} x_{ai} \partial_{bk} \partial_{m+d,j} +$$

$$\sum_{c,d=1}^l Z(u) ((D-m)^{-1})_{cd} \otimes x_{m+c,i} \partial_{m+d,j}$$

which is equal to the sum of the series in the lines (2.8) and (2.9). This equality proves that the map χ intertwines the actions of the Yangian $Y(\mathfrak{gl}_n)$. \square

By the transitivity of induction, Theorem 2.1 can be extended from the maximal to all parabolic subalgebras of the Lie algebra \mathfrak{gl}_m . Consider the case of the Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}'$ of \mathfrak{gl}_m . Here \mathfrak{h} is the Cartan subalgebra of \mathfrak{gl}_m spanned by the elements E_{aa} , whereas \mathfrak{n}' is the nilpotent subalgebra of \mathfrak{gl}_m spanned by the elements E_{ab} with $a < b$.

Take any element μ of the vector space \mathfrak{h}^* dual to \mathfrak{h} , any such element is called a *weight*. The weight μ can be identified with the sequence (μ_1, \dots, μ_m) of its *labels*, where $\mu_a = \mu(E_{aa})$ for each $a = 1, \dots, m$. Consider the Verma module M_μ over the Lie algebra \mathfrak{gl}_m . It can be described as the quotient of the algebra $U(\mathfrak{gl}_m)$ by the left ideal generated by all the elements E_{ab} with $a < b$ and the elements $E_{aa} - \mu_a$. The elements of the Lie algebra \mathfrak{gl}_m act on this quotient via left multiplication. The image of the element $1 \in U(\mathfrak{gl}_m)$ in this quotient is denoted by 1_μ . Then $X \cdot 1_\mu = 0$ for all $X \in \mathfrak{n}'$ while

$$X \cdot 1_\mu = \mu(X) \cdot 1_\mu \quad \text{for all } X \in \mathfrak{h}.$$

Let us apply the functor (1.8) to the \mathfrak{gl}_m -module $V = M_\mu$, and the functor (1.20) to the resulting \mathfrak{H}_N -module

$$W = (M_\mu \otimes (\mathbb{C}^m)^{\otimes N})_{\mathfrak{n}}.$$

We obtain the $Y(\mathfrak{gl}_n)$ -module

$$((M_\mu \otimes (\mathbb{C}^m)^{\otimes N})_{\mathfrak{n}} \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N} = (M_\mu \otimes S^N(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}.$$

By taking the direct sum over $N = 0, 1, 2, \dots$ of these $Y(\mathfrak{gl}_n)$ -modules, we obtain the $Y(\mathfrak{gl}_n)$ -module

$$\mathcal{E}_m(M_\mu)_{\mathfrak{n}} = (M_\mu \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}.$$

Note that $\mathcal{E}_m(M_\mu)_{\mathfrak{n}}$ is also a module over the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$. Using the basis E_{11}, \dots, E_{mm} identify \mathfrak{h} with the direct sum of m copies of the Lie algebra \mathfrak{gl}_1 . Consider the Verma modules $M_{\mu_1}, \dots, M_{\mu_m}$ over \mathfrak{gl}_1 . By applying Theorem 2.1 repeatedly we get the next result, which can be also derived from [AS, Theorem 3.3.1].

Corollary 2.4. *The bimodule $\mathcal{E}_m(M_\mu)_{\mathfrak{n}}$ of \mathfrak{h} and $Y(\mathfrak{gl}_n)$ is equivalent to the tensor product*

$$\mathcal{E}_1(M_{\mu_1}) \otimes \mathcal{E}_1^1(M_{\mu_2}) \otimes \dots \otimes \mathcal{E}_1^{m-1}(M_{\mu_m}).$$

We complete this section with describing for any $t, z \in \mathbb{C}$ the bimodule $\mathcal{E}_1^z(M_t)$ over \mathfrak{gl}_1 and $Y(\mathfrak{gl}_n)$. The Verma module M_t over \mathfrak{gl}_1 is one-dimensional, and the element $E_{11} \in \mathfrak{gl}_1$ acts on M_t as multiplication by t . The vector space of bimodule $\mathcal{E}_1(M_t)$ is the

symmetric algebra $S(\mathbb{C}^1 \otimes \mathbb{C}^n) = S(\mathbb{C}^n)$, which we identify with $\mathcal{P}(\mathbb{C}^1 \otimes \mathbb{C}^n) = \mathcal{P}(\mathbb{C}^n)$. Then E_{11} acts on $\mathcal{E}_1(M_t)$ as the differential operator

$$t + \sum_{k=1}^n x_{1k} \partial_{1k}.$$

The action of E_{11} on $\mathcal{E}_1^z(M_t)$ is the same as on $\mathcal{E}_1(M_t)$. The generator $T_{ij}^{(s+1)}$ of $Y(\mathfrak{gl}_n)$ with $s = 0, 1, 2, \dots$ acts on $\mathcal{E}_1(M_t)$ as the differential operator

$$(-t)^s x_{1i} \partial_{1j},$$

this is what Proposition 1.3 states in the case $m = 1$. Note that the operator $x_{1i} \partial_{1j}$ describes the action on $\mathcal{P}(\mathbb{C}^1 \otimes \mathbb{C}^n) = \mathcal{P}(\mathbb{C}^n)$ of the element $E_{ij} \in \mathfrak{gl}_n$. Hence the action of the algebra $Y(\mathfrak{gl}_n)$ on $\mathcal{E}_1(M_t)$ can be obtained from the action of \mathfrak{gl}_n on $\mathcal{P}(\mathbb{C}^n)$ by pulling back through the homomorphism $\pi_n : Y(\mathfrak{gl}_n) \rightarrow U(\mathfrak{gl}_n)$, and then through the automorphism τ_{-t} of $Y(\mathfrak{gl}_n)$; see the definitions (1.11) and (1.12). Hence the action of $Y(\mathfrak{gl}_n)$ on $\mathcal{E}_1^z(M_t)$ can be obtained from the action of \mathfrak{gl}_n on $\mathcal{P}(\mathbb{C}^n)$ by pulling back through π_n and then through the automorphism τ_{z-t} .

3. Zhelobenko operators

Consider the symmetric group \mathfrak{S}_m as the Weyl group of the reductive Lie algebra \mathfrak{gl}_m . This group acts on the vector space \mathfrak{gl}_m so that for any $\sigma \in \mathfrak{S}_m$ and $a, b = 1, \dots, m$

$$\sigma : E_{ab} \mapsto E_{\sigma(a)\sigma(b)}.$$

This action extends to an action of the group \mathfrak{S}_m by automorphisms of the associative algebra $U(\mathfrak{gl}_m)$. The group \mathfrak{S}_m also acts on the vector space \mathfrak{h}^* . Let $E_{11}^*, \dots, E_{mm}^*$ be the basis of \mathfrak{h}^* dual to the basis E_{11}, \dots, E_{mm} of \mathfrak{h} . Then

$$\sigma : E_{aa}^* \mapsto E_{\sigma(a)\sigma(a)}^*.$$

If we identify each weight $\mu \in \mathfrak{h}^*$ with the sequence (μ_1, \dots, μ_m) of its labels, then

$$\sigma : (\mu_1, \dots, \mu_m) \mapsto (\mu_{\sigma^{-1}(1)}, \dots, \mu_{\sigma^{-1}(m)}).$$

Let $\rho \in \mathfrak{h}^*$ be the weight with sequence of labels $(0, -1, \dots, 1 - m)$. The *shifted* action of any element $\sigma \in \mathfrak{S}_m$ on \mathfrak{h}^* is defined by the assignment

$$\mu \mapsto \sigma \circ \mu = \sigma(\mu + \rho) - \rho. \quad (3.1)$$

For any $a, b = 1, \dots, m$ put $\varepsilon_{ab} = E_{aa}^* - E_{bb}^*$. The elements $\varepsilon_{ab} \in \mathfrak{h}^*$ with $a < b$ and $a > b$ are the *positive* and *negative roots* respectively. Note that $\varepsilon_{ab} = 0$ when $a = b$. The elements $\varepsilon_c = \varepsilon_{c,c+1} \in \mathfrak{h}^*$ with $c = 1, \dots, m - 1$ are the *simple* positive roots. Put

$$E_c = E_{c,c+1}, \quad F_c = E_{c+1,c} \quad \text{and} \quad H_c = E_{cc} - E_{c+1,c+1}.$$

For any $a = 1, \dots, m-1$ these three elements of the Lie algebra \mathfrak{gl}_m span a subalgebra isomorphic to the Lie algebra \mathfrak{sl}_2 .

For any \mathfrak{gl}_m -module V and any $\lambda \in \mathfrak{h}^*$ a vector $v \in V$ is said to be *of weight* λ if $Xv = \lambda(X)v$ for any $X \in \mathfrak{h}$. We will denote by V^λ the subspace in V formed by all vectors of weight λ . Recall that \mathfrak{n} denotes the nilpotent subalgebra of \mathfrak{gl}_m spanned by the elements E_{ab} with $a > b$. In this section, we will employ the general notion of a Mickelsson algebra introduced in [M1] and developed by D. Zhelobenko [Z]. Namely, we will show how this notion gives rise to a distinguished $Y(\mathfrak{gl}_n)$ -intertwining operator

$$\mathcal{E}_m(M_\mu)_\mathfrak{n}^\lambda \rightarrow \mathcal{E}_m(M_{\sigma \circ \mu})_\mathfrak{n}^{\sigma \circ \lambda} \quad (3.2)$$

for any element $\sigma \in \mathfrak{S}_m$ and any weight $\mu \in \mathfrak{h}^*$ such that

$$\mu_a - \mu_b \notin \mathbb{Z} \quad \text{whenever} \quad a \neq b. \quad (3.3)$$

Note that the source and the target vector spaces in (3.2) are non-zero only if all labels of the weight $\lambda - \mu$ are non-negative integers. Then $\lambda_a - \lambda_b \notin \mathbb{Z}$ whenever $a \neq b$.

We have a representation $\gamma : U(\mathfrak{gl}_m) \rightarrow \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ such that the image $\gamma(E_{ab})$ is the differential operator (1.25). Note that the group \mathfrak{S}_m acts by automorphisms of the algebra $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$, so that for $k = 1, \dots, n$

$$\sigma : x_{ak} \mapsto x_{\sigma(a)k}, \quad \partial_{bk} \mapsto \partial_{\sigma(b)k}.$$

The homomorphism γ is \mathfrak{S}_m -equivariant. Let A be the associative algebra generated by the algebras $U(\mathfrak{gl}_m)$ and $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the cross relations

$$[X, Y] = [\gamma(X), Y] \quad (3.4)$$

for any $X \in \mathfrak{gl}_m$ and $Y \in \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Here each pair of the square brackets denotes the commutator in A . Note that the algebra A is isomorphic to the tensor product of associative algebras (0.1). The isomorphism can be defined by mapping the elements $X \in \mathfrak{gl}_m$ and $Y \in \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of the algebra A respectively to the elements

$$X \otimes 1 + 1 \otimes \gamma(X) \quad \text{and} \quad 1 \otimes Y$$

of the algebra (0.1). This isomorphism is \mathfrak{S}_m -equivariant, and the image of the element $E_{ab} \in \mathfrak{gl}_m$ under this isomorphism equals (1.24). We will use this isomorphism later on.

Let J be the right ideal of the algebra A generated by the elements of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$. Let $\text{Norm}(J) \subset A$ be the normalizer of this right ideal, so that $Y \in \text{Norm}(J)$ if and only if $YJ \subset J$. Then J is a two-sided ideal of $\text{Norm}(J)$. Our particular *Mickelsson algebra* is the quotient

$$R = J \setminus \text{Norm}(J). \quad (3.5)$$

Remark. Via its isomorphism with (0.1), the associative algebra A acts on the tensor product $V \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ for any \mathfrak{gl}_m -module V . The defining embedding of \mathfrak{gl}_m into A corresponds to the diagonal action of the Lie algebra \mathfrak{gl}_m on this tensor product. The algebra R then acts on the space of \mathfrak{n} -coinvariants of \mathfrak{gl}_m -module $V \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$. \square

Let $\overline{U(\mathfrak{h})}$ be the ring of fractions of the algebra $U(\mathfrak{h})$ relative to the denominators set

$$\{ E_{aa} - E_{bb} + z \mid 1 \leq a, b \leq m; a \neq b; z \in \mathbb{Z} \}. \quad (3.6)$$

The elements of this ring can also be regarded as rational functions on the vector space \mathfrak{h}^* . The elements of $U(\mathfrak{h}) \subset \overline{U(\mathfrak{h})}$ are then regarded as polynomial functions on \mathfrak{h}^* . Denote by \bar{A} the ring of fractions of A relative to the same set of denominators (3.6), regarded as elements of A using the embedding of $\mathfrak{h} \subset \mathfrak{gl}_m$ into A . The ring \bar{A} is defined due to the following relations in $U(\mathfrak{gl}_m)$ and A : for $a, b = 1, \dots, m$ and any $H \in \mathfrak{h}$

$$[H, E_{ab}] = \varepsilon_{ab}(H) E_{ab}, \quad [H, x_{ak}] = E_{aa}^*(H) x_{ak}, \quad [H, \partial_{bk}] = -E_{bb}^*(H) \partial_{bk}.$$

Therefore the ring A satisfies the Ore condition relative to its subset (3.6). Using left multiplication by elements of $\overline{U(\mathfrak{h})}$, the ring of fractions \bar{A} becomes a module over $\overline{U(\mathfrak{h})}$.

The ring \bar{A} is also an associative algebra over the field \mathbb{C} . For each $c = 1, \dots, m-1$ define a linear map $\xi_c : A \rightarrow \bar{A}$ by setting

$$\xi_c(Y) = Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \hat{F}_c^s(Y) \quad (3.7)$$

for $Y \in A$. Here

$$H_c^{(s)} = H_c(H_c - 1) \dots (H_c - s + 1)$$

and \hat{F}_c is the operator of adjoint action corresponding to the element $F_c \in A$, so that

$$\hat{F}_c(Y) = [F_c, Y].$$

For any given element $Y \in A$ only finitely many terms of the sum (3.7) differ from zero, hence the map ξ_c is well defined. The definition (3.7) and the following proposition go back to [Z, Section 2]. Denote $\bar{J} = \overline{U(\mathfrak{h})} J$. Then \bar{J} is a right ideal of the algebra \bar{A} .

Proposition 3.1. *For any $X \in \mathfrak{h}$ and $Y \in A$ we have*

$$\xi_c(XY) \in (X + \varepsilon_c(X)) \xi_c(Y) + \bar{J}, \quad (3.8)$$

$$\xi_c(YX) \in \xi_c(Y)(X + \varepsilon_c(X)) + \bar{J}. \quad (3.9)$$

Proof. It suffices to verify each of the properties (3.8) and (3.9) for two elements of \mathfrak{h} ,

$$X = E_{cc} + E_{c+1, c+1} \quad \text{and} \quad X = E_{cc} - E_{c+1, c+1} = H_c.$$

For the first of these two elements we have $[E_c, X] = [F_c, X] = 0$ and $\varepsilon_c(X) = 0$, so that the properties (3.8) and (3.9) are obvious.

For $X = H_c$ the proof of (3.8) is based on the following commutation relations in the subalgebra of $U(\mathfrak{gl}_m)$ generated by the three elements E_c, F_c and H_c : for $s = 1, 2, \dots$

$$[E_c^s, H_c] = -2s E_c^s, \quad (3.10)$$

$$[E_c^s, F_c] = s(H_c - s + 1) E_c^{s-1}. \quad (3.11)$$

Let us use the symbol \equiv to indicate equalities in the algebra \bar{A} modulo the ideal \bar{J} . By the definition (3.7), for any element $Y \in A$ we have

$$\begin{aligned}
\xi_c(H_c Y) &= H_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \hat{F}_c^s(H_c Y) \\
&= H_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s (H_c \hat{F}_c^s(Y) + 2s F_c \hat{F}_c^{s-1}(Y)) \\
&\equiv H_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} ((H_c - 2s) E_c^s \hat{F}_c^s(Y) \\
&\quad + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} \cdot 2s^2 (H_c - s + 1) E_c^{s-1} \hat{F}_c^{s-1}(Y)) \\
&= H_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} ((H_c - 2s) E_c^s \hat{F}_c^s(Y) \\
&\quad + \sum_{s=0}^{\infty} (s! H_c^{(s)})^{-1} \cdot 2(s+1) E_c^s \hat{F}_c^s(Y)) \\
&= (H_c + 2) \xi_c(Y) = (H_c + \varepsilon_c(H_c)) \xi_c(Y)
\end{aligned}$$

as needed. To get the equivalence relation above, we also used the inclusion $\overline{U(\mathfrak{h})} F_c \subset \bar{J}$.

Similarly, for any element $Y \in A$ we have

$$\begin{aligned}
\xi_c(Y H_c) &= Y H_c + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \hat{F}_c^s(Y H_c) \\
&= Y H_c + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s (\hat{F}_c^s(Y) H_c + 2s \hat{F}_c^{s-1}(Y) F_c) \\
&= Y H_c + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s (\hat{F}_c^s(Y) H_c - 2s \hat{F}_c^s(Y) + 2s F_c \hat{F}_c^{s-1}(Y)) \\
&\equiv Y H_c + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s (\hat{F}_c^s(Y) H_c - 2s \hat{F}_c^s(Y)) \\
&\quad + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} \cdot 2s^2 (H_c - s + 1) E_c^{s-1} \hat{F}_c^{s-1}(Y) \\
&= Y H_c + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s (\hat{F}_c^s(Y) H_c - 2s \hat{F}_c^s(Y)) \\
&\quad + \sum_{s=0}^{\infty} (s! H_c^{(s)})^{-1} \cdot 2(s+1) E_c^s \hat{F}_c^s(Y) \\
&= \xi_c(Y) (H_c + 2) = \xi_c(Y) (H_c + \varepsilon_c(H_c)). \quad \square
\end{aligned}$$

The property (3.8) allows us to define a linear map $\bar{\xi}_c : \bar{A} \rightarrow \bar{J} \setminus \bar{A}$ by setting

$$\bar{\xi}_c(XY) = Z \xi_c(Y) + \bar{J} \quad \text{for any } X \in \overline{U(\mathfrak{h})} \text{ and } Y \in A,$$

where the element $Z \in \overline{U(\mathfrak{h})}$ is defined by the equality

$$Z(\mu) = X(\mu + \varepsilon_c) \quad \text{for any } \mu \in \mathfrak{h}^*$$

when X and Z are regarded as rational functions on \mathfrak{h}^* .

The action of the group \mathfrak{S}_m on the algebra $U(\mathfrak{h})$ extends to an action on $\overline{U(\mathfrak{h})}$, so that for any $\sigma \in \mathfrak{S}_m$

$$(\sigma X)(\mu) = X(\sigma^{-1}(\mu))$$

when the element $X \in \overline{U(\mathfrak{h})}$ is regarded as a rational function on \mathfrak{h}^* . The action of \mathfrak{S}_m by automorphisms of the algebra A then extends to an action by automorphisms of \bar{A} . For any $c = 1, \dots, m-1$ let $\sigma_c \in \mathfrak{S}_m$ be the transposition of c and $c+1$. Consider the image $\sigma_c(\bar{J})$, this is again a right ideal of \bar{A} . Next proposition also goes back to [Z].

Proposition 3.2. *We have $\sigma_c(\bar{J}) \subset \ker \bar{\xi}_c$.*

Proof. Let \mathfrak{n}_c be the vector subspace in \mathfrak{gl}_m spanned by all the elements E_{ab} where $a > b$ but $(a, b) \neq (c+1, c)$. Then $\sigma_c(\bar{J}) \subset \bar{A}$ is the right ideal generated by the subspace $\mathfrak{n}_c \subset \mathfrak{gl}_m$ and by the element $E_c = E_{c,c+1}$. Here we use the embedding of \mathfrak{gl}_m into \bar{A} . Observe that the subspace $\mathfrak{n}_c \subset \mathfrak{gl}_m$ is preserved by the adjoint action of the elements E_c, F_c and H_c on \mathfrak{gl}_m . Therefore $\xi_c(XY) \in \bar{J}$ for any $X \in \mathfrak{n}_c$ and $Y \in A$, see (3.7). To prove Proposition 3.2 it remains to show that $\xi_c(E_c Y) \in \bar{J}$ for any $Y \in A$. We have

$$\begin{aligned} \xi_c(E_c Y) &= E_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \hat{F}_c^s(E_c Y) \\ &= E_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} E_c^s (E_c \hat{F}_c^s(Y) - s H_c \hat{F}_c^{s-1}(Y)) \\ &\quad - \sum_{s=2}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \cdot s(s-1) F_c \hat{F}_c^{s-2}(Y) \\ &\equiv E_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} (E_c^{s+1} \hat{F}_c^s(Y) - s(H_c - 2s) E_c^s \hat{F}_c^{s-1}(Y)) \\ &\quad - \sum_{s=2}^{\infty} (s! H_c^{(s)})^{-1} s^2 (s-1) (H_c - s + 1) E_c^{s-1} \hat{F}_c^{s-2}(Y) \\ &= E_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} (E_c^{s+1} \hat{F}_c^s(Y) - s(H_c - 2s) E_c^s \hat{F}_c^{s-1}(Y)) \\ &\quad - \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} s(s+1) E_c^s \hat{F}_c^{s-1}(Y) \\ &= E_c Y + \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} (E_c^{s+1} \hat{F}_c^s(Y) - s(H_c - s + 1) E_c^s \hat{F}_c^{s-1}(Y)). \end{aligned}$$

The sum of all terms in the last line is equal to zero. Like in the proof of Proposition 3.1, here we used (3.10), (3.11) and indicated by \equiv the equality in \bar{A} modulo the ideal \bar{J} . \square

Proposition 3.2 allows us to define for any $c = 1, \dots, m-1$ a linear map

$$\check{\xi}_c : \bar{\mathbf{J}} \setminus \bar{\mathbf{A}} \rightarrow \bar{\mathbf{J}} \setminus \bar{\mathbf{A}} \quad (3.12)$$

as the composition $\bar{\xi}_c \sigma_c$ applied to the elements of $\bar{\mathbf{A}}$ which were taken modulo $\bar{\mathbf{J}}$.

Remark. Observe that $U(\mathfrak{h}) \subset \text{Norm}(\mathbf{J})$. Let us denote by $\overline{\text{Norm}(\mathbf{J})}$ the ring of fractions of $\text{Norm}(\mathbf{J})$ relative to the same set of denominators (3.6) as before. Evidently, then $\bar{\mathbf{J}}$ is a two-sided ideal of the ring $\overline{\text{Norm}(\mathbf{J})}$. The quotient ring

$$\bar{\mathbf{R}} = \bar{\mathbf{J}} \setminus \overline{\text{Norm}(\mathbf{J})}$$

bears the same name of Mickelsson algebra, as the quotient ring (3.5) does. One can show [KO] that the linear map (3.12) preserves the subspace $\bar{\mathbf{R}} \subset \bar{\mathbf{J}} \setminus \bar{\mathbf{A}}$, and moreover determines an automorphism of the algebra $\bar{\mathbf{R}}$. Although we do not use these two facts, our construction of the $Y(\mathfrak{gl}_n)$ -intertwining operator (3.2) is underlied by them. \square

In their present form, the operators $\check{\xi}_1, \dots, \check{\xi}_{m-1}$ on the vector space $\bar{\mathbf{J}} \setminus \bar{\mathbf{A}}$ have been introduced in [KO]. We will call them *Zhelobenko operators*. The next proposition states the key property of these operators; for the proof of this proposition see [Z, Section 6].

Proposition 3.3. *The operators $\check{\xi}_1, \dots, \check{\xi}_{m-1}$ on $\bar{\mathbf{J}} \setminus \bar{\mathbf{A}}$ satisfy the braid relations*

$$\begin{aligned} \check{\xi}_c \check{\xi}_{c+1} \check{\xi}_c &= \check{\xi}_{c+1} \check{\xi}_c \check{\xi}_{c+1} & \text{for } c < m-1, \\ \check{\xi}_b \check{\xi}_c &= \check{\xi}_c \check{\xi}_b & \text{for } b < c-1. \end{aligned}$$

Corollary 3.4. *For any reduced decomposition $\sigma = \sigma_{c_1} \dots \sigma_{c_K}$ in \mathfrak{S}_m the composition $\check{\xi}_{c_1} \dots \check{\xi}_{c_K}$ of operators on $\bar{\mathbf{J}} \setminus \bar{\mathbf{A}}$ does not depend on the choice of decomposition of σ .*

Recall that \mathfrak{n}' denotes the nilpotent subalgebra of \mathfrak{gl}_m spanned by the elements E_{ab} with $a < b$. Denote by \mathbf{J}' the left ideal of the algebra \mathbf{A} generated by the elements of the subalgebra $\mathfrak{n}' \subset \mathfrak{gl}_m$. Put $\bar{\mathbf{J}}' = \overline{U(\mathfrak{h})} \mathbf{J}'$. Then $\bar{\mathbf{J}}'$ is a left ideal of $\bar{\mathbf{A}}$. Consider the image $\sigma_c(\bar{\mathbf{J}}')$, this is again a left ideal of $\bar{\mathbf{A}}$.

Proposition 3.5. *We have $\bar{\xi}_c(\sigma_c(\bar{\mathbf{J}}')) \subset \bar{\mathbf{J}}' + \bar{\mathbf{J}}$.*

Proof. Let \mathfrak{n}'_c be the vector subspace in \mathfrak{gl}_m spanned by all elements E_{ab} where $a < b$ but $(a, b) \neq (c, c+1)$. Then $\sigma_c(\bar{\mathbf{J}}') \subset \bar{\mathbf{A}}$ is the left ideal generated by the subspace $\mathfrak{n}'_c \subset \mathfrak{gl}_m$ and by the element $F_c = E_{c+1, c}$. Here we use the embedding of \mathfrak{gl}_m into $\bar{\mathbf{A}}$. Observe that the subspace $\mathfrak{n}'_c \subset \mathfrak{gl}_m$ is preserved by the adjoint action of the element F_c on \mathfrak{gl}_m . Thus $\xi_c(YX) \in \bar{\mathbf{J}}' + \bar{\mathbf{J}}$ for any $X \in \mathfrak{n}'_c$ and $Y \in \mathbf{A}$, see the definition (3.7).

Note that $\xi_c(YF_c) = \xi_c(Y)F_c$ for any $Y \in \mathbf{A}$, because $\widehat{F}_c(YF_c) = \widehat{F}_c(Y)F_c$. We shall complete the proof of Proposition 3.5 by showing that here $\xi_c(Y)F_c \in \bar{\mathbf{J}}$. Indeed,

$$\begin{aligned} \xi_c(Y)F_c &= \sum_{s=0}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \widehat{F}_c^s(Y) F_c = \\ &= \sum_{s=0}^{\infty} (s! H_c^{(s)})^{-1} E_c^s F_c \widehat{F}_c^s(Y) - \sum_{s=0}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \widehat{F}_c^{s+1}(Y) \equiv \end{aligned}$$

$$\begin{aligned} & \sum_{s=1}^{\infty} (s! H_c^{(s)})^{-1} s(H_c - s + 1) E_c^{s-1} \widehat{F}_c^s(Y) - \sum_{s=0}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \widehat{F}_c^{s+1}(Y) = \\ & \sum_{s=1}^{\infty} ((s-1)! H_c^{(s-1)})^{-1} E_c^{s-1} \widehat{F}_c^s(Y) - \sum_{s=0}^{\infty} (s! H_c^{(s)})^{-1} E_c^s \widehat{F}_c^{s+1}(Y) = 0. \end{aligned}$$

Here we used (3.11), and indicated by \equiv an equality in \bar{A} modulo the right ideal \bar{J} . \square

Proposition 3.5 implies that for each $c = 1, \dots, m-1$ the Zhelobenko operator (3.12) induces a linear map

$$\bar{J} \setminus \bar{A} / \bar{J}' \rightarrow \bar{J} \setminus \bar{A} / \bar{J}'.$$

Now take a weight $\mu \in \mathfrak{h}^*$ satisfying (3.3). We shall keep the assumption (3.3) on μ till the end of this section. Let I_μ be the left ideal of the algebra A generated by the elements

$$E_{ab} \quad \text{with } a < b, \quad E_{aa} - \mu_a \quad \text{and} \quad \partial_{bk}$$

for all possible a, b and k . Under the isomorphism of A with the tensor product (0.1), the ideal I_μ of A corresponds to the ideal of the algebra (0.1) generated by the elements

$$E_{ab} \otimes 1 \quad \text{with } a < b, \quad E_{aa} \otimes 1 - \mu_a \quad \text{and} \quad 1 \otimes \partial_{bk}$$

for all possible a, b and k . Indeed, for any $a, b = 1, \dots, m$ the image of the element $E_{ab} \in A$ in the algebra (0.1) is the sum (1.24), which equals $E_{ab} \otimes 1$ plus elements divisible on the right by the tensor products of the form $1 \otimes \partial_{bk}$. But the quotient space of (0.1) with respect to the latter ideal can be naturally identified with the tensor product $M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Using the isomorphism of algebras A and (0.1), the quotient space A/I_μ can be then also identified with $M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$.

Note that $\mu(H_c) \notin \mathbb{Z}$ for any index $c = 1, \dots, m-1$ due to (3.3). Hence we can define the subspace $\bar{I}_\mu = \overline{U(\mathfrak{h})} I_\mu$ of \bar{A} . This subspace is also a left ideal of the algebra \bar{A} . The quotient space \bar{A}/\bar{I}_μ can be still identified with the tensor product $M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$. The quotient of \bar{A} by \bar{I}_μ and \bar{J} can be then identified with the space of \mathfrak{n} -coinvariants,

$$\bar{J} \setminus \bar{A} / \bar{I}_\mu = (M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}. \quad (3.13)$$

Consider the left ideal of the algebra A generated by all the elements ∂_{bk} where $b = 1, \dots, m$ and $k = 1, \dots, n$. By the definition (3.7), the image of this ideal under the map ξ_c is contained in the left ideal of \bar{A} generated by the same elements. The latter ideal is preserved by the action on \bar{A} of the element $\sigma_c \in \mathfrak{S}_m$. Note that by (3.1),

$$\sigma_c(\mu + \varepsilon_c) = \sigma_c \circ \mu.$$

The property (3.9) and Proposition 3.5 now imply that the Zhelobenko operator (3.12) induces a linear map

$$\bar{J} \setminus \bar{A} / \bar{I}_\mu \rightarrow \bar{J} \setminus \bar{A} / \bar{I}_{\sigma_c \circ \mu}.$$

Using the identifications (3.13), the Zhelobenko operator (3.12) induces a linear map

$$(M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}} \rightarrow (M_{\sigma_c \circ \mu} \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}. \quad (3.14)$$

Proposition 3.6. *For any $s = 0, 1, 2, \dots$ the map (3.14) commutes with the action of generator $T_{ij}^{(s+1)}$ of $Y(\mathfrak{gl}_n)$ on the source and target vector spaces as the element (1.22).*

Proof. Let Y be the element of the algebra A corresponding to the element (1.22) of the algebra (0.1) under the isomorphism of these two algebras. The element Y then belongs to the centralizer of the subalgebra $U(\mathfrak{gl}_n)$ in A . So the left multiplication in \bar{A} by Y preserves the right ideal $\bar{J} \subset \bar{A}$, and commutes with the linear map $\xi_c : \bar{A} \rightarrow \bar{J} \setminus \bar{A}$; see the definition (3.7). This left multiplication also commutes with the action of the element $\sigma_c \in \mathfrak{S}_m$ on \bar{A} , because the element Y of the algebra A is \mathfrak{S}_m -invariant. \square

The property (3.8) implies that the restriction of the linear map (3.14) to the subspace of vectors of weight λ is a map

$$(M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}^\lambda \rightarrow (M_{\sigma_c \circ \mu} \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}^{\sigma_c \circ \lambda}.$$

Denote the latter map by I_c , it commutes with the action of $Y(\mathfrak{gl}_n)$ by Proposition 3.6.

By choosing a reduced decomposition $\sigma = \sigma_{c_1} \dots \sigma_{c_K}$ and taking the composition of operators $I_{c_1} \dots I_{c_K}$ we obtain an $Y(\mathfrak{gl}_n)$ -intertwining operator

$$I_\sigma : (M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}^\lambda \rightarrow (M_{\sigma \circ \mu} \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}^{\sigma \circ \lambda}.$$

It does not depend on the choice of the decomposition of $\sigma \in \mathfrak{S}_m$ due to Corollary 3.4. This is the operator (3.2) which we intended to exhibit. Here we identified the symmetric algebra $S(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ in the same way as we did in Section 1.

From now on we will assume that all labels of the weight $\nu = \lambda - \mu$ are non-negative integers, otherwise both the source and target modules in (3.2) are zero. Let (ν_1, \dots, ν_m) be the sequence of these labels. Consider the vector

$$1_\mu \otimes x_{11}^{\nu_1} \dots x_{m1}^{\nu_m} \in M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n) \quad (3.15)$$

Note that $x_{11}^{\nu_1} \dots x_{m1}^{\nu_m}$ is a *highest* vector with respect to the action of the Lie algebra \mathfrak{gl}_n on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$, any element $E_{ij} \in \mathfrak{gl}_n$ with $i < j$ acts on this vector as zero. With respect to the action of the Lie algebra \mathfrak{gl}_m the vector (3.15) is of weight λ . Denote by v_μ^λ the image of the vector (3.15) in $(M_\mu \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}^\lambda$.

Proposition 3.7. *Under condition (3.3), the vector $I_\sigma(v_\mu^\lambda)$ equals $v_{\sigma \circ \mu}^{\sigma \circ \lambda}$ multiplied by*

$$\prod_{\substack{1 \leq a < b \leq m \\ \sigma(a) > \sigma(b)}} \prod_{r=1}^{\nu_b} \frac{\mu_a - \mu_b - a + b - r}{\lambda_a - \lambda_b - a + b + r}. \quad (3.16)$$

Proof. It suffices to prove the proposition for $\sigma = \sigma_c$ with $c = 1, \dots, m-1$. Moreover, it then suffices to consider only the case when $m = 2$ and hence $c = 1$. Suppose this is the case. Then under the identification (3.13) the vector

$$v_\mu^\lambda \in (M_\mu \otimes \mathcal{P}(\mathbb{C}^2 \otimes \mathbb{C}^n))_{\mathfrak{n}}$$

gets identified with the image in the quotient space $\bar{J} \setminus \bar{A} / \bar{I}_\mu$ of the element

$$x_{11}^{\nu_1} x_{21}^{\nu_2} \in A \subset \bar{A}.$$

By applying the transposition $\sigma_1 \in \mathfrak{S}_2$ to this element we get $x_{11}^{\nu_2} x_{21}^{\nu_1} \in A$. By applying the operator ξ_1 to the latter element we get the sum of elements of \bar{A} ,

$$\sum_{s=0}^{\infty} (s! H_1^{(s)})^{-1} E_1^s \hat{F}_1^s(x_{11}^{\nu_2} x_{21}^{\nu_1}). \quad (3.17)$$

In particular, here by the definition of the algebra A we have

$$\hat{F}_1(x_{11}^{\nu_2} x_{21}^{\nu_1}) = [F_1, x_{11}^{\nu_2} x_{21}^{\nu_1}] = \sum_{k=1}^n [x_{2k} \partial_{1k}, x_{11}^{\nu_2} x_{21}^{\nu_1}] = \nu_2 \cdot x_{11}^{\nu_2-1} x_{21}^{\nu_1+1}.$$

Thus the sum (3.17) equals

$$\sum_{s=0}^{\nu_2} (s! H_1^{(s)})^{-1} E_1^s \cdot \prod_{r=1}^s (\nu_2 - r + 1) \cdot x_{11}^{\nu_2-s} x_{21}^{\nu_1+s}. \quad (3.18)$$

Since the element $E_1 \in A$ is a generator of the left ideal $I_{\sigma_1 \circ \mu}$, the image of the sum (3.18) in the quotient vector space $\bar{J} \setminus \bar{A} / \bar{I}_{\sigma_1 \circ \mu}$ coincides with that of

$$\sum_{s=0}^{\nu_2} (s! H_1^{(s)})^{-1} \cdot \prod_{r=1}^s (\nu_2 - r + 1) \cdot \hat{E}_1^s(x_{11}^{\nu_2-s} x_{21}^{\nu_1+s}). \quad (3.19)$$

Here \hat{E}_1 is the operator of adjoint action on $\mathcal{PD}(\mathbb{C}^2 \otimes \mathbb{C}^n)$ corresponding to the element $E_1 \in A$. By (3.4) this operator coincides with the operator of adjoint action of

$$\gamma(E_1) = \sum_{k=1}^n x_{1k} \partial_{2k}.$$

Therefore the sum (3.19) of elements of the algebra \bar{A} equals

$$\sum_{s=0}^{\nu_2} (s! H_1^{(s)})^{-1} \cdot \prod_{r=1}^s (\nu_1 + r) (\nu_2 - r + 1) \cdot x_{11}^{\nu_2} x_{21}^{\nu_1}. \quad (3.20)$$

In the sum (3.20) the symbol $H_1^{(s)}$ stands for the product in the algebra A ,

$$\prod_{r=1}^s (H_1 - r + 1) = \prod_{r=1}^s (E_{11} - E_{22} - r + 1).$$

The operator of adjoint action on $\mathcal{PD}(\mathbb{C}^2 \otimes \mathbb{C}^n)$ corresponding to the element $H_1 \in A$ coincides with the operator of adjoint action of the element

$$\gamma(H_1) = \sum_{k=1}^n (x_{1k} \partial_{1k} - x_{2k} \partial_{2k})$$

The latter operator acts on $x_{11}^{\nu_2} x_{21}^{\nu_1}$ as the multiplication by $\nu_2 - \nu_1$. Since the elements

$$E_{11} - \mu_2 + 1 \quad \text{and} \quad E_{22} - \mu_1 - 1$$

are also generators of the left ideal $I_{\sigma_1 \circ \mu}$, the image of the sum (3.20) in $\bar{J} \setminus \bar{A} / \bar{I}_{\sigma_1 \circ \mu}$ coincides with the image of

$$\sum_{s=0}^{\nu_2} \prod_{r=1}^s \frac{(\nu_1 + r)(\nu_2 - r + 1)}{r(\mu_2 - \mu_1 + \nu_2 - \nu_1 - r - 1)} \cdot x_{11}^{\nu_2} x_{21}^{\nu_1}.$$

The latter image in the quotient vector space $\bar{J} \setminus \bar{A} / \bar{I}_{\sigma_1 \circ \mu}$ is identified with the vector

$$v_{\sigma_1 \circ \mu}^{\sigma_1 \circ \lambda} \in (M_{\sigma_1 \circ \mu} \otimes \mathcal{P}(\mathbb{C}^2 \otimes \mathbb{C}^n))_{\mathbf{n}}$$

multiplied by the sum

$$\sum_{s=0}^{\nu_2} \prod_{r=1}^s \frac{(\nu_1 + r)(\nu_2 - r + 1)}{r(\mu_2 - \mu_1 + \nu_2 - \nu_1 - r - 1)} = \prod_{r=1}^{\nu_2} \frac{\mu_1 - \mu_2 - r + 1}{\lambda_1 - \lambda_2 + r + 1}. \quad (3.21)$$

The equality (3.21) has been also used in [TV1, Theorem 8]. Here is its direct proof. Recall that

$$\mu_1 - \mu_2 + \nu_1 - \nu_2 = \lambda_1 - \lambda_2.$$

Let us write $\nu_1 = x$, $\nu_2 = d$ and $\lambda_1 - \lambda_2 = t$. Then the equality (3.21) takes the form

$$\sum_{s=0}^d \prod_{r=1}^s \frac{(x+r)(d-r+1)}{r(-t-r-1)} = \prod_{r=1}^d \frac{t-x+d-r+1}{t+r+1}. \quad (3.22)$$

We shall prove the latter equality for all $x, t \in \mathbb{C}$ and all nonnegative integers d . Define a family of polynomials $\varphi_{d,t}(x)$ of degree d in the variable x with coefficients from the field $\mathbb{C}(t)$ by the following conditions:

$$\begin{aligned} \varphi_{0,t}(x) &= 1, \\ \varphi_{d,t}(x-1) - \varphi_{d,t}(x) &= d(t+2)^{-1} \varphi_{d-1,t+1}(x) \quad \text{for } d \geq 1, \\ \varphi_{d,t}(-1) &= 1. \end{aligned}$$

These conditions uniquely determine the polynomials $\varphi_{d,t}(x)$. One can easily check that both sides of the equality (3.22) satisfy these conditions. This proves the equality. \square

Note that the $Y(\mathfrak{gl}_n)$ -intertwining operator I_σ has been defined only when the weight μ satisfies the condition (3.3). We also assume that all labels of the weight $\nu = \lambda - \mu$ are non-negative integers. Recall that the sequence of labels (ρ_1, \dots, ρ_m) of the weight ρ is $(0, \dots, 1-m)$. For any $t \in \mathbb{C}$ and $N = 0, 1, 2, \dots$ let us denote by S_t^N the $Y(\mathfrak{gl}_n)$ -module obtained from the standard action of $U(\mathfrak{gl}_n)$ on $S^N(\mathbb{C}^n)$ by pulling back through the homomorphism $\pi_n : Y(\mathfrak{gl}_n) \rightarrow U(\mathfrak{gl}_n)$, and then through the automorphism τ_{-t} of $Y(\mathfrak{gl}_n)$; see the definitions (1.11) and (1.12). Using Corollary 2.4 and the subsequent

remarks, we can replace the source and target modules in (3.2) by equivalent $Y(\mathfrak{gl}_n)$ -modules to get an intertwining operator between two tensor products of $Y(\mathfrak{gl}_n)$ -modules,

$$S_{\mu_1+\rho_1}^{\nu_1} \otimes \dots \otimes S_{\mu_m+\rho_m}^{\nu_m} \rightarrow S_{\tilde{\mu}_1+\tilde{\rho}_1}^{\tilde{\nu}_1} \otimes \dots \otimes S_{\tilde{\mu}_m+\tilde{\rho}_m}^{\tilde{\nu}_m} \quad (3.23)$$

where we write

$$\tilde{\mu}_a = \mu_{\sigma^{-1}(a)}, \quad \tilde{\nu}_a = \nu_{\sigma^{-1}(a)}, \quad \tilde{\rho}_a = \rho_{\sigma^{-1}(a)}$$

for any $a = 1, \dots, m$. It is well known that under the condition (3.3) on the sequence (μ_1, \dots, μ_m) both tensor products are irreducible $Y(\mathfrak{gl}_n)$ -modules, equivalent to each other; see for instance [NT2, Theorem 3.4]. So an intertwining operator between these two tensor products is unique up to a multiplier from \mathbb{C} . For the intertwining operator corresponding to I_σ this multiplier is determined by Proposition 3.7. Another expression for an intertwining operator between two tensor products of $Y(\mathfrak{gl}_n)$ -modules (3.23) can be obtained by using a method of I. Cherednik [C1], see for instance [NT2, Section 2].

Remark. The product (3.16) in Proposition 3.7 does not depend on the choice of reduced decomposition $\sigma_{c_1} \dots \sigma_{c_K}$ of the element $\sigma \in \mathfrak{S}_m$. The uniqueness of the intertwining operator (3.23) thus provides another proof of the independence of the composition $I_{c_1} \dots I_{c_K}$ on the choice of the decomposition of σ , not involving Corollary 3.4. \square

A connection between the intertwining operators on the m -fold tensor products of $Y(\mathfrak{gl}_n)$ -modules of the form S_w^N , and the results of [Z] for the Lie algebra \mathfrak{gl}_m has been established by V. Tarasov and A. Varchenko [TV2]. The construction of the operator I_σ given in this section provides a representation theoretic explanation of that connection.

4. Olshanski homomorphism

Let l be a positive integer. The decomposition $\mathbb{C}^{n+l} = \mathbb{C}^n \oplus \mathbb{C}^l$ defines an embedding of the direct sum $\mathfrak{gl}_n \oplus \mathfrak{gl}_l$ of Lie algebras into \mathfrak{gl}_{n+l} . As a subalgebra of \mathfrak{gl}_{n+l} , the direct summand \mathfrak{gl}_n is spanned by the matrix units $E_{ij} \in \mathfrak{gl}_{n+l}$ where $i, j = 1, \dots, n$. The direct summand \mathfrak{gl}_l is spanned by the matrix units E_{ij} where $i, j = n+1, \dots, n+l$. Let C_l be the centralizer in $U(\mathfrak{gl}_{n+l})$ of the subalgebra $\mathfrak{gl}_l \subset \mathfrak{gl}_{n+l}$. Set $C_0 = U(\mathfrak{gl}_n)$.

Proposition 1.3 shows that for any positive integer m a homomorphism of associative algebras

$$Y(\mathfrak{gl}_n) \rightarrow U(\mathfrak{gl}_m) \otimes \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n) \quad (4.1)$$

can be defined by mapping $T_{ij}^{(s+1)}$ to the sum (1.22). The image of this homomorphism is contained in the centralizer of the image of \mathfrak{gl}_m in $U(\mathfrak{gl}_m) \otimes \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$, see the remark just after the proof of Proposition 1.3. In this final section we will compare this homomorphism with a homomorphism $Y(\mathfrak{gl}_n) \rightarrow C_l$ defined by G. Olshanski [O1].

Consider the Yangian $Y(\mathfrak{gl}_{n+l})$. The subalgebra in $Y(\mathfrak{gl}_{n+l})$ generated by

$$T_{ij}^{(1)}, T_{ij}^{(2)}, \dots \quad \text{where } i, j = 1, \dots, n$$

is isomorphic to $Y(\mathfrak{gl}_n)$ as an associative algebra, see [MNO, Corollary 1.23]. Thus we have a natural embedding $Y(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_{n+l})$, which will be denoted by ι_l . Note that ι_l is not a Hopf algebra homomorphism. We also have a surjective homomorphism

$$\pi_{n+l} : Y(\mathfrak{gl}_{n+l}) \rightarrow U(\mathfrak{gl}_{n+l}),$$

see (1.12). The composition $\pi_{n+l} \iota_l$ coincides with the homomorphism π_n .

Further, consider the involutive automorphism ω_{n+l} of the algebra $Y(\mathfrak{gl}_{n+l})$, see the definition (2.2). The image of the composition of homomorphisms

$$\pi_{n+l} \omega_{n+l} \iota_l : Y(\mathfrak{gl}_n) \rightarrow U(\mathfrak{gl}_{n+l})$$

belongs to subalgebra $C_l \subset U(\mathfrak{gl}_{n+l})$. Moreover, this image along with the centre of the algebra $U(\mathfrak{gl}_{n+l})$ generates the subalgebra C_l . For the proofs of these two assertions see [O2, Section 2.1]. The *Olshanski homomorphism* $Y(\mathfrak{gl}_n) \rightarrow C_l$ is the composition

$$\alpha_l = \pi_{n+l} \omega_{n+l} \iota_l \tau_{-l}. \quad (4.2)$$

Set $\alpha_0 = \pi_n$. The family of homomorphisms $\alpha_0, \alpha_1, \alpha_2, \dots$ has the following property of stability [O1]. Let us denote by I the left ideal of the algebra $U(\mathfrak{gl}_{n+l})$ generated by the elements $E_{1,n+l}, \dots, E_{n+l,n+l}$. One can easily check that the intersection $I \cap C_l$ is a two-sided ideal of the algebra C_l . Moreover, there is a decomposition

$$C_l = C_{l-1} \oplus (I \cap C_l). \quad (4.3)$$

For the proofs of these two assertions again see [O2, Section 2.1]. In the equality (4.3) we regard $C_{l-1} \subset U(\mathfrak{gl}_{n+l-1})$ as a subalgebra in $U(\mathfrak{gl}_{n+l})$ using the standard embedding

$$U(\mathfrak{gl}_{n+l-1}) \rightarrow U(\mathfrak{gl}_{n+l}).$$

Denote by ϖ_l the projection to the first direct summand in (4.3). Because $I \cap C_l$ is a two-sided ideal of C_l , the linear map $\varpi_l : C_l \rightarrow C_{l-1}$ is an algebra homomorphism.

Proposition 4.1. *For any $l = 1, 2, \dots$ we have $\varpi_l \alpha_l = \alpha_{l-1}$.*

Proof. See [O2, Section 2.1] once again. \square

Let us describe the homomorphism α_l more explicitly. For any $i, j = 1, \dots, n+l$ regard E_{ij} as an element of the algebra $U(\mathfrak{gl}_{n+l})$. Consider the $(n+l) \times (n+l)$ matrix whose ij -entry is $\delta_{ij} - E_{ij} u^{-1}$. Consider the inverse of this matrix as a formal power series in u^{-1} with matrix coefficients. Denote by $Y_{ij}(u)$ the ij -entry of the inverse matrix,

$$Y_{ij}(u) = \delta_{ij} + E_{ij} u^{-1} + \sum_{s=1}^{\infty} \sum_{k_1, \dots, k_s=1}^{n+l} E_{ik_1} E_{k_1 k_2} \dots E_{k_{s-1} k_s} E_{k_s j} u^{-s-1}.$$

Then by (4.2),

$$\alpha_l : T_{ij}(u) \mapsto Y_{ij}(u+l) \quad \text{for } i, j = 1, \dots, n. \quad (4.4)$$

Here each of the formal power series $Y_{ij}(u+l)$ in $(u+l)^{-1}$ should be re-expanded in u^{-1} , and (4.4) is a correspondence between the respective coefficients of series in u^{-1} .

In the present article, we will employ the homomorphism $Y(\mathfrak{gl}_n) \rightarrow C_l$

$$\beta_l = \alpha_l \omega_n = \pi_{n+l} \omega_{n+l} \iota_l \omega_n \tau_l.$$

The second equality here follows from the definition (4.2) and from the relation

$$\tau_{-l} \omega_n = \omega_n \tau_l,$$

see (1.11) and (2.2). The image of any series (1.9) under β_l can be expressed in terms of quasideterminants [BK, Lemma 4.2] or quantum minors [BK, Lemma 8.5]; see also [NT1, Lemma 1.5]. The reason for considering here the homomorphism β_l rather than α_l will become clear when we state Proposition 4.3. Using the definitions of β_l and β_{l-1} only, Proposition 4.1 can be restated as

Corollary 4.2. *For any $l = 1, 2, \dots$ we have $\varpi_l \beta_l = \beta_{l-1}$.*

Consider the Lie algebra \mathfrak{gl}_m and its Cartan subalgebra \mathfrak{h} . A weight $\mu \in \mathfrak{h}^*$ is called *polynomial* if its labels μ_1, \dots, μ_m are non-negative integers such that $\mu_1 \geq \dots \geq \mu_m$. The weight $\mu \in \mathfrak{h}^*$ is polynomial if and only if for some non-negative integer N the vector space

$$\text{Hom}_{\mathfrak{gl}_m}(L_\mu, (\mathbb{C}^m)^{\otimes N}) \neq \{0\}.$$

Then

$$N = \mu_1 + \dots + \mu_m.$$

The irreducible \mathfrak{gl}_m -module L_μ of highest weight μ is then called a *polynomial* module. Then by setting $\mu_{m+1} = \mu_{m+2} = \dots = 0$ we get a partition (μ_1, μ_2, \dots) of N . When there is no confusion with the polynomial weight of \mathfrak{gl}_m , this partition will be denoted by μ as well. The maximal index a with $\mu_a > 0$ is then called the *length* of the partition, and is denoted by $\ell(\mu)$. Note that here $\ell(\mu) \leq m$. Further, let $\mu^* = (\mu_1^*, \mu_2^*, \dots)$ be the partition *conjugate* to the partition μ . By definition, here μ_b^* is equal to the maximal index a such that $\mu_a \geq b$. In particular, here $\mu_1^* = \ell(\mu)$.

Let λ and μ two polynomial weights of \mathfrak{gl}_m such that

$$\ell(\lambda) \leq n + l \quad \text{and} \quad \ell(\mu) \leq l. \quad (4.5)$$

Using the respective partitions, then λ and μ can also be regarded as polynomial weights of the Lie algebras \mathfrak{gl}_{n+l} and \mathfrak{gl}_l respectively. Denote by L'_λ and L'_μ the corresponding irreducible highest weight modules. We use this notation to distinguish them from the irreducible modules L_λ and L_μ over the Lie algebra \mathfrak{gl}_m .

Using the action of the Lie algebra \mathfrak{gl}_l on L'_λ via its embedding into \mathfrak{gl}_{n+l} as the second direct summand of the subalgebra $\mathfrak{gl}_n \oplus \mathfrak{gl}_l \subset \mathfrak{gl}_{n+l}$ consider the vector space

$$\text{Hom}_{\mathfrak{gl}_l}(L'_\mu, L'_\lambda). \quad (4.6)$$

The subalgebra $C_l \subset U(\mathfrak{gl}_{n+l})$ acts on this vector space through the action of $U(\mathfrak{gl}_{n+l})$ on L'_λ . Moreover, the action of C_l on (4.6) is irreducible [D, Theorem 9.1.12]. Hence the following identifications of C_l -modules are unique up to rescaling of the vector spaces:

$$\begin{aligned} & \text{Hom}_{\mathfrak{gl}_l}(L'_\mu, L'_\lambda) = \\ & \text{Hom}_{\mathfrak{gl}_l}(L'_\mu, \text{Hom}_{\mathfrak{gl}_m}(L_\lambda, S(\mathbb{C}^m \otimes \mathbb{C}^{n+l}))) = \\ & \text{Hom}_{\mathfrak{gl}_l}(L'_\mu, \text{Hom}_{\mathfrak{gl}_m}(L_\lambda, S(\mathbb{C}^m \otimes \mathbb{C}^l) \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n))) = \\ & \text{Hom}_{\mathfrak{gl}_m}(L_\lambda, L_\mu \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n)). \end{aligned} \quad (4.7)$$

Here we use the classical identifications of modules over the Lie algebras \mathfrak{gl}_{n+l} and \mathfrak{gl}_m ,

$$L'_\lambda = \text{Hom}_{\mathfrak{gl}_m}(L_\lambda, S(\mathbb{C}^m \otimes \mathbb{C}^{n+l}))$$

and

$$\text{Hom}_{\mathfrak{gl}_l}(L'_\mu, S(\mathbb{C}^m \otimes \mathbb{C}^l)) = L_\mu$$

respectively, see for instance [H, Section 2.1]. We also use the decomposition

$$S(\mathbb{C}^m \otimes \mathbb{C}^{n+l}) = S(\mathbb{C}^m \otimes \mathbb{C}^l) \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n).$$

By pulling back through the homomorphism $\beta_l : Y(\mathfrak{gl}_n) \rightarrow C_l$, the vector space (4.7) becomes a module over the Yangian $Y(\mathfrak{gl}_n)$. On the other hand, the target vector space $L_\mu \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n)$ in (4.7) coincides with the vector space of the bimodule $\mathcal{E}_m(L_\mu)$ over \mathfrak{gl}_m and $Y(\mathfrak{gl}_n)$, see the beginning of Section 2. Using this bimodule structure, the vector space (4.7) becomes another module over $Y(\mathfrak{gl}_n)$. The next proposition shows that these two $Y(\mathfrak{gl}_n)$ -modules are the same. It also makes [A, Remark 12] more precise. We will give a direct proof, another proof can be obtained by using [BK, Lemma 4.2].

Proposition 4.3. *The action of the algebra $Y(\mathfrak{gl}_n)$ on the vector space (4.7) via the homomorphism β_l coincides with the action inherited from the bimodule $\mathcal{E}_m(L_\mu)$.*

Proof. Consider the action of the subalgebra $C_l \subset U(\mathfrak{gl}_{n+l})$ on $S(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$. The Yangian $Y(\mathfrak{gl}_n)$ acts on this vector space via the homomorphism $\beta_l : Y(\mathfrak{gl}_n) \rightarrow C_l$. Let us identify this vector space with $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$ so that the standard basis vectors of $\mathbb{C}^m \otimes \mathbb{C}^{n+l}$ are identified with the corresponding coordinate functions x_{ai} where $a = 1, \dots, m$ and $i = 1, \dots, n+l$. Using the decomposition

$$\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^{n+l}) = \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^l) \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n) \quad (4.8)$$

we will demonstrate that for any $s = 0, 1, 2, \dots$ and $i, j = 1, \dots, n$ the generator $T_{ij}^{(s+1)}$ of $Y(\mathfrak{gl}_n)$ then acts on the vector space (4.8) as the element (1.22) of the tensor product $U(\mathfrak{gl}_m) \otimes \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Proposition 4.3 will thus follow from Proposition 1.3.

For any $i, j = 1, \dots, n+l$ the element $E_{ij} \in U(\mathfrak{gl}_{n+l})$ acts on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$ as the differential operator

$$\sum_{c=1}^m x_{ci} \partial_{cj}.$$

Consider the $(n+l) \times (n+l)$ matrix whose ij -entry is

$$\delta_{ij} + (u-l)^{-1} \sum_{c=1}^m x_{ci} \partial_{cj}.$$

Write this matrix and its inverse as the block matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$$

where the blocks A, B, C, D and $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are matrices of sizes $n \times n, n \times l, l \times n, l \times l$ respectively. The action of the algebra $Y(\mathfrak{gl}_n)$ on the vector space $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$ via the homomorphism $\beta_l : Y(\mathfrak{gl}_n) \rightarrow C_l$ can now be described by assigning to the series $T_{ij}(u)$ with $i, j = 1, \dots, n$ the ij -entry of the matrix \tilde{A}^{-1} .

Consider the $(n+l) \times m$ matrix whose ic -entry is the operator of multiplication by x_{ci} in $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$. Write it as

$$\begin{bmatrix} P \\ \tilde{P} \end{bmatrix}$$

where the blocks P and \bar{P} are matrices of sizes $n \times m$ and $l \times m$ respectively. Similarly, consider the $m \times (n + l)$ matrix whose cj -entry is the operator ∂_{cj} . Write this matrix as

$$\begin{bmatrix} Q & \bar{Q} \end{bmatrix}$$

where the blocks Q and \bar{Q} are matrices of sizes $m \times n$ and $m \times l$ respectively. Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1 + (u - l)^{-1} \begin{bmatrix} P \\ \bar{P} \end{bmatrix} \begin{bmatrix} Q & \bar{Q} \end{bmatrix} = 1 + (u - l)^{-1} \begin{bmatrix} PQ & P\bar{Q} \\ \bar{P}Q & \bar{P}\bar{Q} \end{bmatrix}$$

where 1 stands for the $(n + l) \times (n + l)$ identity matrix. By using Lemma 2.2, we get

$$\begin{aligned} \tilde{A}^{-1} &= A - BD^{-1}C = \\ &= 1 + (u - l)^{-1}PQ - (u - l)^{-2}P\bar{Q}(1 + (u - l)^{-1}\bar{P}\bar{Q})^{-1}\bar{P}Q = \\ &= 1 + P(u - l + \bar{Q}\bar{P})^{-1}Q \end{aligned} \tag{4.9}$$

where 1 now stands for the $n \times n$ identity matrix.

Consider the $m \times m$ matrix $u - l + \bar{Q}\bar{P}$ appearing in the line (4.9). Its ab -entry is

$$\delta_{ab}(u - l) + \sum_{k=1}^l \partial_{a,n+k} x_{b,n+k} = \delta_{ab}u + \sum_{k=1}^l x_{b,n+k} \partial_{a,n+k}.$$

Observe that the last displayed sum over $k = 1, \dots, l$ corresponds to the action of the element $E_{ba} \in \mathcal{U}(\mathfrak{gl}_m)$ on the first tensor factor in the decomposition (4.8). Denote by $Z_{ab}(u)$ the ab -entry of the matrix inverse to $u - l + \bar{Q}\bar{P}$. The ij -entry of the matrix (4.9) can then be written as the sum

$$\delta_{ij} + \sum_{a,b=1}^m x_{ai} Z_{ab}(u) \partial_{bj} = \delta_{ij} + \sum_{a,b=1}^m Z_{ab}(u) x_{ai} \partial_{bj}.$$

Using the observation made in the end of Section 1, we now complete the proof. \square

Note that the proof of Proposition 4.3 remains valid in the case $l = 0$. In this case we assume that $\mathfrak{gl}_l = \{0\}$. Further note that the homomorphism $\mathcal{U}(\mathfrak{gl}_m) \rightarrow \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^l)$ corresponding to the action of \mathfrak{gl}_m on the first tensor factor in the decomposition (4.8) is injective if $l \geq m$. Thus, independently of Proposition 1.3, our proof of Proposition 4.3 shows that for any positive integer m a homomorphism of associative algebras (4.1) can be defined by mapping $T_{ij}^{(s+1)}$ to the sum (1.22).

Now for any given partitions λ and μ consider all integers l satisfying the conditions (4.5). Then consider the corresponding $\mathcal{Y}(\mathfrak{gl}_n)$ -modules (4.6) where the integers l vary. By choosing a positive integer m such that $\ell(\lambda), \ell(\mu) \leq m$ we derive from Proposition 4.3 the following known fact, cf. [N2, Theorem 1.6].

Corollary 4.4. *For all integers l obeying the conditions (4.5) the $Y(\mathfrak{gl}_n)$ -modules (4.6) are equivalent.*

Further, for any given polynomial weights λ and μ of \mathfrak{gl}_m we can choose an integer l large enough to satisfy the conditions (4.5). Then the algebra C_l acts on the vector space (4.7) irreducibly, while the central elements of $U(\mathfrak{gl}_{n+l})$ act on (4.7) via multiplication by scalars. Hence Proposition 4.3 implies another known fact; cf. [A, Theorem 10].

Corollary 4.5. *The action of the algebra $Y(\mathfrak{gl}_n)$ on the vector space (4.7) inherited from the bimodule $\mathcal{E}_m(L_\mu)$ is irreducible for any polynomial weights λ and μ of \mathfrak{gl}_m .*

Furthermore, it is well known that the vector space (4.6) is not zero if and only if

$$\lambda_a \geq \mu_a \quad \text{and} \quad \lambda_a^* - \mu_a^* \leq n \quad \text{for every} \quad a = 1, \dots, m. \quad (4.10)$$

By choosing, for given polynomial weights λ and μ of \mathfrak{gl}_m , an integer l satisfying (4.5), and then identifying the vector spaces (4.6) and (4.7), we get another well known fact.

Corollary 4.6. *The space (4.7) is not zero if and only if the inequalities (4.10) hold.*

For further details on the irreducible representations of the Yangian $Y(\mathfrak{gl}_n)$ of the form (4.6) see for instance [M, Section 4] and [NT1, Section 2].

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